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THE BEHAVIOR OF A COMPOSITE DISC IN PLANE ELASTIC EQUILIBRIUM

TECHNICAL REPORT

BY

STEPHEN D. BECK

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77

CONTENTS

LIST OF TABLES

LIST OF FIGURES

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ABSTRACT

This investigation is directed to the analysis of a composite disc in plane elastic equilibrium. The disc consists of concentric regions, each of homogeneous properties, and is loaded on its outer surface by an arbitrary load. Reactions at the center are prescribed to provide the necessary equilibrating force and moment. The equations of equilibrium are specified in terms of the radial and tangential displacements, giving a pair of partial differential in plane polar coordinates.

A finite Fourier transform is applied to reduce the partial differential equations to a pair of ordinary differential equations with the radial coordinate as independent variables. Inversion of the transformed solution yields an infinite series. The analysis is continued in further detail for uniformly distributed and concentrated normal and shear loads.

The deformations associated with the uniformly distributed normal load are used as the basis of a postulated friction phenomenon called the deformation coefficient of friction. Assumptions about the behavior of the strain energy as a load is translated an incremental distance give rise to a friction force, which in turn permits a friction coefficient to be calculated. This coefficient is characterized by being proportional to the load rather than constant. The contribution of the deformation coefficient to the total friction behavior is ordinarily quite small.

PART I.

INTRODUCTION AND HISTORICAL REVIEW

A. Introduction

This thesis is concerned with a problem in the plane theory of elasticity. We shall consider a composite disc consisting of two perfectly bonded, coplaner, and concentric regions which are each isotropic and homogeneous. The disc is loaded by arbitrary stresses on the outer boundary. A concentrated central force and moment equilibrate the boundary tractions. A formal solution in the form of an infinite series is obtained for the general case. Solutions are obtained explicitly for a few special loadings, and numerical values obtained for the particular case of a normal stress of unit intensity distributed over a segment of the boundary.

A friction mechanism is postulated, based on the strain energy stored in the system during an incremental translation of the area of application of the normal stress. This deformation coefficient of friction depends on the elastic behavior of a smooth disc under the influence of a normal load. It does not take into consideration other possible sources of friction, such as sticking and the interaction of asperities. The deformation coefficient is ordinarily quite small compared to Columb friction, and has the unusual property of being proportional to the load. The frictional force contributed by this mechanism, therefore, varies as the square of the normal load.

The mathematical analysis used is somewhat unusual. The equilibrium equations are established in terms of displacements. A

finite Fourier exponential transform is used to reduce the system of partial differential equations to ordinary differential equations. The transformed solution is inverted by an infinite series. The transform method has properties which make it quite useful in circular regions. Unlike the more familiar Kolosoff-Muskhelishvili method, in which complex potentials of known form are required, the transform method leads to the solution through a direct process. The resulting infinite series corresponds to the series required to establish the harmonic functions in a composite region when the Kolosoff-Muskhelishvili method is used. In the particular case when the disc is homogeneous the series solutions may be exhibited in closed form. When the distributed load is condensed to a concentrated load the solution agrees with previously known solutions.

B. Historical Review

The analysis of circular regions in a state of plane elastic equilibrium has been the subject of substantial investigation. One of the earliest reports on the subject was that of Michell[22*] in which he examined a class of problems in the plane disc. Michell considered concentrated loads in the interior of an infinite plane and on the edge of a plane half-space. By superposition of these effects he developed formulas for the disc loaded at the center and edge by concentrated loads. He gave explicit formulas for the stress field due to concentrated loads at the edge with an equilibrating reaction at the center.

*Throughout this thesis, superscript numbers refer to the similarly numbered items in PART VII, BIBLIOGRAPHY.

These formulas are unfortunately incorrect due to failure to properly annihilate the reactions on the boundary due to the edge load.

Mindlin[25] extended Michell's problem to consider the effect of translating the central force to an arbitrary interior point, directed along a diameter. As a starting point he used Michell's problem, for which he expresses, without derivation, the correct Airy stress function.

The development of the Kolosoff-Muskhelishvili[27, 32] complex variable method made it possible to solve a large class of problems in the disc. In most cases it is necessary to use a catalog of harmonic functions to satisfy the boundary conditions. Mikhlin[23] examined a composite disc loaded on the boundary by self-equilibrating forces. His solution is in the form of infinite series of harmonic functions.

Dokos[9] examined a composite consisting of a disc loaded at the center and embedded in an infinite plate. His method consisted of superposition of field equations for stress and displacement. Dundurs and Hetenyi[11, 16, 10] examined a series of problems of the elastic disc embedded in an infinite plate. The Kolosoff-Muskhelishvili formulas were used somewhat in their development, but the method was basically a superposition of appropriate Airy stress functions.

Ng[28] resolved Michell's problem using the complex variable method. His results, given in terms of displacements, can be shown to be in agreement with those of Mindlin.

A recent investigation by Maye and Ling[20] used finite Fourier transforms, along with the theory of couple stresses, to examine the ring. Maye developed the equations for a composite disc as a special case.

Due to the complexities associated with couple stresses an explicit analytical solution was not given. His technique of inversion uses only the real part of the series.

Finite Fourier transforms have been developed by Doetsch[8] and Kniess[18]. Applications of these transforms to various vibration and heat conduction problems have been developed by Brown[2, 3, 4] and Roettinger[30]. Generally these investigations have been restricted to the sine or cosine transform.

PART II.

THEORY

A. Formulation of the Problem

Consider a circular region in a state of plane elastic equilibrium. The equations of equilibrium in cylindrical coordinates may be written in terms of the radial and circumferential displacements

$$(k+1)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{k+1}{r}\frac{\partial u}{\partial r} - \frac{k+1}{r^2}u + \frac{k}{r}\frac{\partial^2 v}{\partial r\,\partial \theta} - \frac{k+2}{r^2}\frac{\partial v}{\partial \theta} = 0$$

$$\frac{k}{r}\frac{\partial^2 u}{\partial r\,\partial \theta} + \frac{k+2}{r^2}\frac{\partial u}{\partial \theta} + \frac{\partial^2 v}{\partial r^2} + \frac{k+1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}v = 0 \qquad (1)$$

For a composite, as in this analysis, Eqs. (1) must hold in both regions. Constants and variables associated with the core will be distinguished by a bar (e.g. $\bar{u}$), while the unbarred symbols will apply to the ring, or to the whole system, as appropriate.

The elastic constant k is related to the more familiar Lamé constants and Poisson's ratio, in the case of plane strain, by

$$k = (\lambda+\mu)/\mu = 1/(1-2\nu) \qquad (2)$$

and for plane stress by

$$k = (3\lambda + 2\mu)/\mu = (1+\nu)/(1-\nu) \tag{3}$$

Stresses in each region may be expressed in terms of the displacements by

$$\begin{aligned}
\sigma_r &= \mu \left[(k+1)\frac{\partial u}{\partial r} + \frac{k-1}{r}\left(u + \frac{\partial v}{\partial \theta}\right) \right] \\
\sigma_\theta &= \mu \left[(k-1)\frac{\partial u}{\partial r} + \frac{k+1}{r}\left(u + \frac{\partial v}{\partial \theta}\right) \right] \\
\tau_{r\theta} &= \mu \left[\frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial \theta} - v\right) \right]
\end{aligned} \tag{4}$$

We shall take as our dimensions a disc whose outer radius is unity and for which the radius of the interface is δ. Prescribed surface tractions on the outer circle are equilibrated by concentrated force and moment at the center. The polar coordinates (r, θ) of a generic point are measured from the center of the disc and the x axis. Figure 1 is a representation of the elastic system.

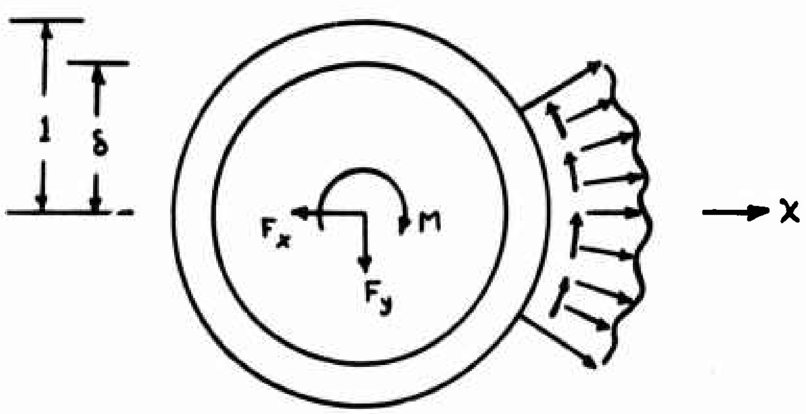

Figure 1. Composite Elastic Disc

It is convenient to separate the boundary stress into its normal and tangential components. Therefore we express the conditions at $r = 1$ by

$$\sigma_r(1,\theta) = g(\theta)$$
$$\tau_{r\theta}(1,\theta) = f(\theta) \tag{5}$$

In order to establish the conditions at the center of the disc we consider the requirement that the concentrated central force and moment equilibrate the force and moment resultant of the boundary stresses. The force will be compactly expressed in complex form terms of its x and y components. The force and moment resultants are then given by

$$F_x + i F_y = \int_{-\pi}^{\pi} \left[g(\theta) + i f(\theta)\right] e^{i\theta} d\theta$$
$$M = \int_{-\pi}^{\pi} f(\theta)\, d\theta \tag{6}$$

We will first look at a central hole of radius ϵ. Consider the surface of the hole to be acted on by a constant shearing stress $\bar{f}/\epsilon^2$ and by a normal stress $\bar{g}(\theta)/\epsilon$. The boundary conditions associated with these requirements are given by

$$\bar{\sigma}_r(\epsilon,\theta) = \bar{g}(\theta)/\epsilon$$
$$\bar{\tau}_{r\theta}(\epsilon,\theta) = \bar{f}/\epsilon^2 \tag{7}$$

The condition of equilibrium with the surface tractions requires that

$$\int_{-\pi}^{\pi} \bar{g}(\theta)\, e^{i\theta}\, d\theta = F_x + i F_y$$
$$\int_{-\pi}^{\pi} \bar{f}\, d\theta = M \qquad\qquad \Rightarrow \bar{f} = M/2\pi \tag{8}$$

The condition of concentrated force and moment at the center is obtained from the limit as $\epsilon \rightarrow 0$.

The boundary problem is completely prescribed when we add the conditions that u, v, σ_r, and $\tau_{r\theta}$ are continuous at the interface between the regions.

Eqs. (1), (4), (5), and the limit of (7), subject to the side condition (8) and the matching conditions at the interface, establish a boundary value problem for the disc. We shall examine the problem in the general case and then look at some particular examples in detail.

B. Transformation of the Problem

The solution of the posed boundary value problem will be obtained by means of the finite Fourier transform. Our definition, which is similar to that of Doetsch[8], is

$$Z_s = \left\{ Z(\theta) \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(\theta)\, e^{-is\theta}\, d\theta \tag{9}$$

whose inverse is

$$Z(\theta) = \sum_{s=-\infty}^{\infty} Z_s \, e^{is\theta} \tag{10}$$

The correspondence of Eqs. (9) and (10) is well known from the theory of Fourier series[7]. Except for the fundamental analysis by Doetsch and the recent work of Maye[20], however, the use of Eq. (9) as a transform is rare. The corresponding Fourier sine and cosine transforms have been used somewhat more frequently for the solution of initial and boundary value problems[2, 3, 4, 30]. To obtain the transforms of derivatives we apply the definition to the n th derivative of $Z(\theta)$. Integrate once by parts, to obtain

$$\left\{Z^{(n)}(\theta)\right\} = is\left\{Z^{(n-1)}(\theta)\right\} - \frac{(-1)^s}{2\pi}\left[Z^{(n-1)}(\pi) - Z^{(n-1)}(-\pi)\right] \tag{11}$$

In a circular region $\theta = \pi$ and $\theta = -\pi$ coincide. Therefore if $Z(\theta)$ and its first $n-1$ derivatives are continuous, the constant part of Eq. (11) vanishes. Transforms of any derivative of $Z(\theta)$ can then be written in terms of Z_s as

$$\left\{Z^{(n)}(\theta)\right\} = (is)^n \left\{Z(\theta)\right\} = (is)^n Z_s \tag{12}$$

The power of the finite Fourier transforms in circular regions is

enhanced by this simplification. For any function which is reasonably well behaved it is possible to orient the coordinates to assure the required continuity. In the analysis which follows it is assumed that Eq. (12) holds.

The transformed equilibrium conditions Eqs. (1) are

$$
\begin{aligned}
&(k+1)\, u_s'' + \frac{k+1}{r}\, u_s' - \frac{k+s^2+1}{r^2}\, u_s \\
&\qquad + i\left[\frac{ks}{r}\, v_s' - \frac{(k+2)s}{r^2}\, v_s\right] = 0 \\
&\frac{ks}{r}\, u_s' + \frac{(k+2)s}{r^2}\, u_s \\
&\qquad - i\left[v_s'' + \frac{1}{r}\, v_s' - \frac{(k+1)s^2+1}{r^2}\, v_s\right] = 0
\end{aligned}
\tag{13}
$$

This coupled pair of ordinary differential equations may be solved by elementary methods, giving for $|s| \geq 2$

$$
\begin{aligned}
u_s &= U_{s1}\, r^{s+1} + U_{s2}\, r^{-(s+1)} + U_{s3}\, r^{s-1} + U_{s4}\, r^{-(s-1)} \\
v_s &= i\Big[\frac{ks+2k+2}{ks-2}\, U_{s1}\, r^{s+1} - U_{s2}\, r^{-(s+1)} \\
&\qquad + U_{s3}\, r^{s-1} - \frac{ks-2k-2}{ks+2}\, U_{s4}\, r^{-(s-1)}\Big]
\end{aligned}
\tag{14}
$$

For $s = 0$ the equations become uncoupled,

$$
\begin{aligned}
u_0'' + \frac{1}{r} u_0' - \frac{1}{r^2} u_0 &= 0 \\
v_0'' + \frac{1}{r} v_0' - \frac{1}{r^2} v_0 &= 0
\end{aligned}
\tag{15}
$$

whose solution is

$$
\begin{aligned}
u_0 &= U_{01} r + U_{02} r^{-1} \\
v_0 &= V_{01} r + V_{02} r^{-1}
\end{aligned}
\tag{16}
$$

When $|s| = 1$ the solution given by Eqs. (14) is incomplete because of a repeated function. However, if we rewrite Eqs. (13) for $s = \pm 1$

$$
\begin{aligned}
(k+1) u_s'' + \frac{k+1}{r} u_s' - \frac{k+2}{r^2} u_s + is\left[\frac{k}{r} v_s' - \frac{k+2}{r^2} v_s\right] &= 0 \\
\frac{k}{r} u_s' + \frac{k+2}{r^2} u_s - is\left[v_s'' + \frac{1}{r} v_s' - \frac{k+2}{r^2} v_s\right] &= 0
\end{aligned}
\tag{17}
$$

we obtain the solution

$$
\begin{aligned}
u_s &= U_{s1} r^2 + U_{s2} r^{-2} + U_{s3} + U_{s4} \log r \\
v_s &= is\left[\frac{3k+2}{k-2} U_{s1} r^2 - U_{s2} r^{-2} + U_{s3} \right. \\
&\qquad \left. + \frac{k}{k+2} U_{s4} + U_{s4} \log r\right]
\end{aligned}
\tag{18}
$$

The transformed stresses are expressed in terms of the transformed displacements

$$\sigma_{rs} = \mu\left[(k+1)u_s' + \frac{k-1}{r}(u_s + is v_s)\right]$$

$$\sigma_{\theta s} = \mu\left[(k-1)u_s' + \frac{k+1}{r}(u_s + is v_s)\right] \tag{19}$$

$$\tau_s = -i\mu\left[i v_s' - \frac{1}{r}(s u_s + i v_s)\right]$$

More explicitly, for $s = 0$

$$\sigma_{r0} = 2\mu(k U_{01} - U_{02} r^{-2})$$

$$\sigma_{\theta 0} = 2\mu(k U_{01} + U_{02} r^{-2}) \tag{20}$$

$$\tau_0 = -2\mu V_{02} r^{-2}$$

When $s = \pm 1$

$$\sigma_{rs} = -2\mu\left[\frac{2k}{k-2}U_{s1} r + 2U_{s2} r^{-3} - \frac{2k+1}{k+2}U_{s4} r^{-1}\right]$$

$$\sigma_{\theta s} = -2\mu\left[\frac{6k}{k-2}U_{s1} r - 2U_{s2} r^{-3} + \frac{1}{k+2}U_{s4} r^{-1}\right] \tag{21}$$

$$\tau_s = 2is\mu\left[\frac{2k}{k-2}U_{s1} r + 2U_{s2} r^{-3} + \frac{1}{k+2}U_{s4} r^{-1}\right]$$

For $s \geq 2$

$$\sigma_{rs} = 2\mu\left[\frac{k(s+1)(s-2)}{ks-2} U_{s1} r^{s} - (s+1) U_{s2} r^{-(s+2)} + (s-1) U_{s3} r^{s-2} - \frac{k(s-1)(s+2)}{ks+2} U_{s4} r^{-s}\right]$$

$$\sigma_{\theta s} = -2\mu\left[\frac{k(s+1)(s+2)}{ks-2} U_{s1} r^{s} - (s+1) U_{s2} r^{-(s+2)} + (s-1) U_{s3} r^{s-2} - \frac{k(s-1)(s-2)}{ks+2} U_{s4} r^{-s}\right] \tag{22}$$

$$\tau_s = 2i\mu\left[\frac{ks(s+1)}{ks-2} U_{s1} r^{s} + (s+1) U_{s2} r^{-(s+2)} + (s-1) U_{s3} r^{s-2} + \frac{ks(s-1)}{ks+2} U_{s4} r^{-s}\right]$$

C. Solution and Inversion

The solution of the disc problem requires the determination of eight constants for each value of S, four in each of the ring and the central core. The eight boundary conditions to be satisfied are

$$\sigma_{rs}(1) - g_s = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, e^{-is\theta}\, d\theta \qquad \text{(a)}$$

(23)

$$\tau_s(1) = f_s = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-is\theta}\, d\theta \qquad \text{(b)}$$

$$\sigma_{rs}(\delta) - \bar{\sigma}_{rs}(\delta) = 0 \qquad \text{(c)}$$

$$\tau_s(\delta) - \bar{\tau}_s(\delta) = 0 \qquad \text{(d)}$$

$$u_s(\delta) - \bar{u}_s(\delta) = 0 \qquad \text{(e)}$$

(23) Cont.

$$v_s(\delta) - \bar{v}_s(\delta) = 0 \qquad \text{(f)}$$

$$\bar{\sigma}_{rs}(\epsilon) = \bar{g}_s\,\epsilon^{-1} = \frac{\epsilon^{-1}}{2\pi}\int_{-\pi}^{\pi} \bar{g}(\theta)\,e^{-is\theta}\,d\theta \qquad \text{(g)}$$

$$\bar{\tau}_{rs}(\epsilon) = \bar{f}_s\,\epsilon^{-2} = \frac{\bar{f}\epsilon^{-2}}{2\pi}\int_{-\pi}^{\pi} e^{-is\theta}\,d\theta \qquad \text{(h)}$$

where (g) and (h) are to be satisfied in the limit. Evaluating the integral in (h) gives

$$\bar{f}_s = \begin{cases} M/2\pi & (s = 0) \\ 0 & (s = 0) \end{cases} \qquad (24)$$

We will examine the boundary conditions for three classes of values of s, that is, for $s = 0$, $|s| = 1$, and $|s| \geq 2$.

If we express the boundary conditions explicitly for $s = 0$ we can write for the V_{0j} constants

$$V_{02} = -f_0/2\mu \qquad \text{(b)}$$

(25)

$$\mu V_{02} - \bar{\mu}\,\bar{V}_{02} = 0 \qquad \text{(d)}$$

$$\delta V_{01} + \delta^{-1} V_{02} - \delta \bar{V}_{01} - \delta^{-1} \bar{V}_{02} = 0 \qquad \text{(f)}$$

(25)
Cont.

$$\epsilon^{-2} \bar{V}_{02} = \bar{f}_0 \epsilon^{-2}/2\bar{\mu} \qquad \text{(h)}$$

where

$$f_0 = \bar{f}_0 = M/2\pi \qquad (26)$$

Conditions (b) and (h) are redundant. As a result, one constant (we choose V_{01}) must remain undetermined. The undetermined constant corresponds to a rigid rotation of the system. The constants are found to be

$$V_{02} = -M/4\pi\mu$$

$$\bar{V}_{02} = -M/4\pi\bar{\mu} \qquad (27)$$

$$\bar{V}_{01} = M(\mu - \bar{\mu})\delta^{-2}/4\pi\mu\bar{\mu} + V_{01}$$

The boundary conditions for the U_{0j} coefficients are given explicitly by

$$k U_{01} - U_{02} = g_0/2\mu \qquad \text{(a)}$$

(28)

$$\mu k U_{01} - \mu \delta^{-2} U_{02} - \bar{\mu} \bar{k} \bar{U}_{01} + \bar{\mu} \delta^{-2} \bar{U}_{02} = 0 \qquad \text{(c)}$$

$$\delta U_{01} + \delta^{-1} U_{02} - \delta \bar{U}_{01} - \delta^{-1} \bar{U}_{02} = 0 \qquad \text{(e)}$$

(28) Cont.

$$\bar{k} \bar{U}_{01} - \epsilon^{-2} \bar{U}_{02} = g_0 \epsilon^{-1} / 2\bar{\mu} \qquad \text{(g)}$$

Application of the limiting condition to (g) shows that

$$\bar{U}_{02} = 0 \tag{29}$$

The remaining conditions, expressed in matrix form are

$$\begin{bmatrix} \mu k & -\mu & 0 \\ \mu k \delta^2 & -\mu & -\bar{\mu} \bar{k} \delta^2 \\ \delta^2 & 1 & -\delta^2 \end{bmatrix} \begin{bmatrix} U_{01} \\ U_{02} \\ \bar{U}_{01} \end{bmatrix} = \frac{g_0}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

from which we find

$$U_{01} = \frac{g_0(\mu + \bar{\mu}\bar{k})}{2\mu D_0}$$

$$U_{02} = \frac{g_0(\mu k - \bar{\mu}\bar{k})\delta^2}{2\mu D_0} \tag{31}$$

$$\bar{U}_{01} = \frac{g_0(k+1)}{2 D_0}$$

where

$$D_0 = \mu k(1 - \delta^2) + \bar{\mu}\bar{k}(k + \delta^2)$$

Consider next the conditions for $|S| = 1$. Substituting Eqs. (18) and (21) into Eqs. (23) gives the explicit conditions

$$\frac{2k}{k-2}U_{s1} + 2U_{s2} - \frac{2s+1}{k+2}U_{s4} = -g_s/2\mu \quad \text{(a)}$$

$$\frac{2k}{k-2}U_{s1} + 2U_{s2} + \frac{1}{k+2}U_{s4} = -i s f_s/2\mu \quad \text{(b)}$$

$$\frac{2\mu k\delta}{k-2}U_{s1} + 2\mu\delta^{-3}U_{s2} - \frac{\mu(2k+1)\delta^{-1}}{k+2}U_{s4} - \frac{2\bar{\mu}\bar{k}\delta}{\bar{k}-2}\bar{U}_{s1} - 2\bar{\mu}\delta^{-3}\bar{U}_{s2} + \frac{\bar{\mu}(2\bar{k}+1)\delta^{-1}}{\bar{k}+2}\bar{U}_{s4} = 0 \quad \text{(c)}$$

$$\frac{2\mu k\delta}{k-2}U_{s1} + 2\mu\delta^{-3}U_{s2} + \frac{\mu\delta^{-1}}{k+2}U_{s4} - \frac{2\bar{\mu}\bar{k}\delta}{\bar{k}-2}\bar{U}_{s1} - 2\bar{\mu}\delta^{-3}\bar{U}_{s2} - \frac{\bar{\mu}\delta^{-1}}{\bar{k}+2}\bar{U}_{s4} = 0 \quad \text{(d)} \qquad (32)$$

$$\delta^2 U_{s1} + \delta^{-2}U_{s2} + U_{s3} + (\log\delta)U_{s4} - \delta^2\bar{U}_{s1} - \delta^{-2}\bar{U}_{s2} - \bar{U}_{s3} - (\log\delta)\bar{U}_{s4} = 0 \quad \text{(e)}$$

$$\frac{(3k+2)\delta^2}{k-2}U_{s1} - \delta^{-2}U_{s2} + U_{s3} + \frac{k}{k+2}U_{s4} + (\log\delta)U_{s4} - \frac{(3\bar{k}+2)\delta^2}{\bar{k}-2}\bar{U}_{s1} + \delta^{-2}\bar{U}_{s2} - \bar{U}_{s3} - \frac{\bar{k}}{\bar{k}+2}\bar{U}_{s4} - (\log\delta)\bar{U}_{s4} = 0 \quad \text{(f)}$$

$$\frac{2\bar{k}\epsilon}{\bar{k}-2}\bar{U}_{s1} + 2\epsilon^{-3}\bar{U}_{s2} - \frac{(2\bar{k}+1)\epsilon^{-1}}{\bar{k}+2}\bar{U}_{s4} = -\bar{g}_s\epsilon^{-1}/2\bar{\mu} \tag{g}$$

(32) Cont.

$$\frac{2\bar{k}\epsilon}{\bar{k}-2}\bar{U}_{s1} + 2\epsilon^{-3}\bar{U}_{s2} + \frac{\epsilon^{-1}}{\bar{k}+2}\bar{U}_{s4} = 0 \tag{h}$$

From Eqs. (6), (8), and (9) we find

$$\bar{g}_s = g_s - isf_s \tag{33}$$

Combining Eqs. (32a) and (b) gives U_{s4}; (g) and (h) give $\bar{U}_{s4}$. Taking (g) or (h) in the limit as $\epsilon \to 0$ gives $\bar{U}_{s2}$. We obtain, thereby

$$\begin{aligned} \bar{U}_{s2} &= 0 \\ U_{s4} &= \frac{(k+2)(g_s - isf_s)}{4\mu(k+1)} \\ \bar{U}_{s4} &= \frac{(\bar{k}+2)(g_s - isf_s)}{4\bar{\mu}(\bar{k}+1)} \end{aligned} \tag{34}$$

Elimination of these constants from Eqs. (32) leaves five equations in five unknown constants. The constants U_{s3} and $\bar{U}_{s3}$ appear only in combination. We can eliminate their difference by combining (e) and (f). The remaining four equations have three unknown quantities. The equations, however, are redundant. One may therefore be eliminated. We may write the remaining three equations in compact form. Let us define

$$X_{s1} = \frac{4U_{s1}}{k-2} \qquad X_{s2} = 4U_{s2} \qquad X_{s3} = \frac{4\bar{U}_{s1}}{\bar{k}-2}$$

$$Q_{s1} = -\frac{g_s}{2\mu\bar{\mu}(k+1)(\bar{k}+1)} \qquad Q_{s2} = \frac{isf_s}{2\mu\bar{\mu}(k+1)(\bar{k}+1)} \tag{35}$$

$$\gamma_1 = 1 \qquad \gamma_2 = -(2k+1)$$

We may now express the remaining equations in matrix form.

$$\begin{bmatrix} k & 1 & 0 \\ \mu k\delta^4 & \mu & -\bar{\mu}\bar{k}\delta^4 \\ (k+2)\delta^4 & -1 & -(\bar{k}+2)\delta^4 \end{bmatrix} \begin{bmatrix} X_{s1} \\ X_{s2} \\ X_{s3} \end{bmatrix} = \sum_{j=1}^{2} Q_{sj} \begin{bmatrix} \gamma_j \bar{\mu}(\bar{k}+1) \\ \delta^2 \mu\bar{\mu}(\bar{k}-k) \\ \delta^2[\bar{\mu}k(\bar{k}+1)-\mu\bar{k}(k+1)] \end{bmatrix} \tag{36}$$

The solution to Eqs. (36) is

$$U_{s1} = \frac{\bar{\mu}(\bar{k}+1)(k-2)}{4D_1} \sum_{j=1}^{2} \left\{ \mu\left[\gamma_j(\bar{k}+2) - (2\bar{k} + k\bar{k} - 2k)\delta^2\right] + \bar{\mu}\bar{k}(\gamma_j + k\delta^2) \right\} Q_{sj} \tag{37}$$

$$U_{s2} = \frac{\bar{\mu}(\bar{k}+1)\delta^2}{4D_1} \sum_{j=1}^{2} \left\{ \mu k\left[2\bar{k} + k\bar{k} - 2k - \gamma_j(\bar{k}+2)\delta^2\right] - \bar{\mu}\bar{k}\left[k^2 - \gamma_j(k+2)\delta^2\right] \right\} Q_{sj}$$

$$\bar{U}_{s1} = \frac{\mu(k+1)(\bar{k}-2)}{4D_1\delta^2}\sum_{j=1}^{2}\Big\{\mu k\bar{k}(1-\delta^4)$$

$$-\bar{\mu}\left[k\bar{k} - 2\gamma_j(\bar{k}+1)\delta^2 - (2k+k\bar{k}-2\bar{k})\delta^4\right]\Big\}Q_{sj} \qquad \text{(37) Cont.}$$

where

$$D_1 = \mu k(\bar{k}+2)(1-\delta^4) + \bar{\mu}\bar{k}(k+k\delta^4+2\delta^4)$$

One constant remains arbitrary. (We choose U_{s3}). The remaining constant is then given by

$$\bar{U}_{s3} = U_{s3} + (U_{s1}-\bar{U}_{s1})\delta^2 + U_{s2}\delta^{-2} + (U_{s4}-\bar{U}_{s4})\log\delta \qquad (38)$$

We have expressed the solution for $s=\pm 1$. The two terms in the solution for each of the constants correspond to the normal and shear components of the boundary tractions.

Let us next examine the constants for $|s| \geq 2$. We observe first that if we use Eqs. (22) to explicitly state boundary conditions given by Eqs. (23g) and (h), we obtain relations of the form

$$\bar{U}_{s2} = a_1\epsilon^{2(s+1)}\bar{U}_{s1} + a_2\epsilon^{2s}\bar{U}_{s3} + a_3\epsilon^{s+1}\bar{g}_s$$

$$\bar{U}_{s4} = b_1\epsilon^{2s}\bar{U}_{s1} + b_2\epsilon^{2(s-1)}\bar{U}_{s3} + b_3\epsilon^{s-1}\bar{g}_s \qquad (39)$$

where a_j and b_j represent collections of constants. If we take the limit as $\epsilon\rightarrow 0$ for $s \geq 2$, we find

$$\bar{U}_{s2} = \bar{U}_{s4} = 0 \qquad (s\geq 2) \qquad (40)$$

Corresponding statements can be written for $s \leq -2$, from which we find

$$\bar{U}_{s3} = \bar{U}_{s1} = 0 \qquad (s \leq -2) \tag{41}$$

As in the case of the indices $s = 0$ and $s = \pm 1$, it does not matter how the normal stress is distributed at the center.

For ease in handling the equations, it will be convenient to express the indices in terms of the positive integer t, defined

$$t = |s| \tag{42}$$

We can now take advantage of the symmetry of form for positive and negative indices by defining

$$\begin{aligned}
[X_{t1} \quad X_{-t1}] &= \frac{\mu k(t+1)}{kt-2} [U_{t1} \quad U_{-t4}] \\
[X_{t2} \quad X_{-t2}] &= \mu(t+1) [U_{t2} \quad U_{-t3}] \\
[X_{t3} \quad X_{-t3}] &= \mu(t-1) [U_{t3} \quad U_{-t2}] \\
[X_{t4} \quad X_{-t4}] &= \frac{\mu k(t-1)}{kt+2} [U_{t4} \quad U_{-t1}] \\
[X_{t5} \quad X_{-t5}] &= \frac{\bar{\mu} \bar{k}(t+1)}{\bar{k} t-2} [\bar{U}_{t1} \quad \bar{U}_{-t4}] \\
[X_{t6} \quad X_{-t6}] &= \bar{\mu}(t-1) [\bar{U}_{t3} \quad \bar{U}_{-t2}]
\end{aligned} \tag{43}$$

We also define

$$Q_{s1} = g_s/4$$

$$Q_{s2} = -i\,(s/t)\,f_s/4 \qquad (44)$$

$$\lambda_1 = -1 \qquad\qquad \lambda_2 = 1$$

If we recombine the remaining six conditions of Eqs. (23) in pairs we may write the boundary conditions explicitly and compactly in matrix form, as is shown in Eq. (45).

$$
\begin{bmatrix}
t-1 & 0 & 1 & -1 & 0 & 0 \\
1 & 1 & 0 & t+1 & 0 & 0 \\
(t-1)\delta^{2t} & 0 & \delta^{2(t-1)} & -1 & -(t-1)\delta^{2t} & -\delta^{2(t-1)} \\
\delta^{2(t+1)} & 1 & 0 & (t+1)\delta^{2} & -\delta^{2(t+1)} & 0 \\
\bar{\mu}k(t-1)\delta^{2t} & 0 & \bar{\mu}k\delta^{2(t-1)} & \bar{\mu}(k+2) & -\mu k(t-1)\delta^{2t} & -\mu k\delta^{2(t-1)} \\
\bar{\mu}\bar{k}(k+2)\delta^{2(t+1)} & -\bar{\mu}\bar{k}k & 0 & -\bar{\mu}\bar{k}k(t+1)\delta^{2} & -\mu k(\bar{k}+2)\delta^{2(t+1)} & 0
\end{bmatrix}
\begin{bmatrix} X_{s1} \\ X_{s2} \\ X_{s3} \\ X_{s4} \\ X_{s5} \\ X_{s6} \end{bmatrix}
= \sum_{j=1}^{2} Q_{sj}
\begin{bmatrix} 1 \\ \lambda_j \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\tag{45}
$$

The solution of Eq. (45) may be written in relatively compact form if we define the following collections of terms:

$$
\begin{aligned}
W &= \bar{\mu}\bar{k}(k+2) - \mu k(\bar{k}+2) \\
X &= \mu k - \bar{\mu}\bar{k} \\
Y &= \mu - \bar{\mu} \\
Z &= \bar{\mu}(k+2) + \mu k \\
\bar{Z} &= \mu(\bar{k}+2) + \bar{\mu}\bar{k}
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
D_t = &-k Z\bar{Z} + t^2 k^2 Y\bar{Z}\,\delta^{2(t-1)} \\
&-2(t^2-1)k^2 Y\bar{Z}\,\delta^{2t} \\
&+\left[t^2 k^2 Y\bar{Z} + 4\bar{\mu}(k+1)X\right]\delta^{2(t+1)} + kYW\delta^{4t}
\end{aligned}
$$

The solution, then, is

$$
\begin{aligned}
X_{s1} = \frac{1}{D_t}\sum_{j=1}^{2} Q_{sj}\Big\{&\lambda_j k Z\bar{Z} - (t+1+\lambda_j)k^2 Y\bar{Z}\,\delta^{2(t-1)} \\
&+(t+1)k^2 Y\bar{Z}\,\delta^{2t}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
X_{s2} = -\frac{\delta^{2t}}{D_t}\sum_{j=1}^{2} Q_{sj}\Big\{&(t+1)\left[\lambda_j(t-1)-1\right]k^2 Y\bar{Z} \\
&-\lambda_j\left[t^2 k^2 Y\bar{Z} + 4\bar{\mu}(k+1)X\right]\delta^2 \\
&-(t+1+\lambda_j)kYW\delta^{2t}\Big\}
\end{aligned}
\tag{47}
$$

$$X_{s3} = \frac{1}{D_t}\sum_{j=1}^{2} Q_{sj}\Big\{\big[\lambda_j(t-1)-1\big]k Z\bar{Z}$$

$$-(t-1)(t+1+\lambda_j)k^2 Y\bar{Z}\,\delta^{2t}$$

$$+\big[t^2k^2Y\bar{Z}+4\bar{\mu}(k+1)X\big]\delta^{2(t+1)}\Big\}$$

$$X_{s4} = \frac{\delta^{2(t-1)}}{D_t}\sum_{j=1}^{2} Q_{sj}\Big\{\big[\lambda_j(t-1)-1\big]k^2 Y\bar{Z}$$

$$-\lambda_j(t-1)k^2Y\bar{Z}\,\delta^2 - kYW\delta^{2(t+1)}\Big\}$$

$$X_{s5} = -\frac{2\bar{\mu}\bar{k}(k+1)}{D_t}\sum_{j=1}^{2} Q_{sj}\Big\{\lambda_j Z \tag{47 Cont.}$$

$$-(t+1+\lambda_j)kY\delta^{2(t-1)}+(t+1)kY\delta^{2t}\Big\}$$

$$X_{s6} = \frac{2\bar{\mu}(k+1)}{D_t}\sum_{j=1}^{2} Q_{sj}\Big\{\big[\lambda_j(t-1)-1\big]k\bar{Z}$$

$$-2\lambda_j(t-1)X\delta^2 - (t-1)(t+1+\lambda_j)k\bar{k}Y\delta^{2t}$$

$$+\big[t^2k\bar{k}Y+2X\big]\delta^{2(t+1)}\Big\}$$

The solution is now formally complete. The coefficients have all been calculated, and may be summed to give the desired functions. The inversion formula, Eq. (10), is somewhat more manageable if broken up into sine and cosine series. Inversion of the results will therefore be effected using

$$Z(r,\theta) = Z_0(r) + \sum_{s=1}^{\infty} \left\{ \left[Z_s(r) + Z_{-s}(r) \right] \cos(s\theta) + i \left[Z_s(r) - Z_{-s}(r) \right] \sin(s\theta) \right\} \tag{48}$$

PART III.

EXAMPLES

A. Uniform and Concentrated Normal Loads

As the first example we shall look at the composite disc loaded in its plane by a unit normal stress over the arc $(-\Phi/2, \Phi/2)$. The boundary conditions may be explicitly expressed

$$\sigma_r(1,\theta) = g(\theta) = \begin{cases} 1 & (-\frac{\Phi}{2} < \theta < \frac{\Phi}{2}) \\ 0 & \text{elsewhere} \end{cases} \tag{49}$$

$$\tau_{r\theta}(1,\theta) = f(\theta) = 0$$

The corresponding conditions for the transformed problem are

$$g_s = \begin{cases} \Phi/2\pi & (s=0) \\ \sin(s\Phi/2)/\pi s & (s \neq 0) \end{cases} \tag{50}$$

$$f_s = 0$$

A limiting case of the distributed normal load is the concentrated unit normal force. To obtain this case from the distributed load, we first divide by $2\sin(\Phi/2)$ to normalize the force. As Φ vanishes the load approaches a concentrated force.

The boundary conditions for the natural problem can be written

$$\sigma_r(1,\theta) = g\ (\theta) = \begin{cases} \lim\limits_{\Phi \to 0} \dfrac{1}{2\sin(\Phi/2)} & (-\frac{\Phi}{2} < \theta < \frac{\Phi}{2}) \\ 0 & \text{elsewhere} \end{cases} \tag{51}$$

The transformed condition for the concentrated normal force is therefore

$$g_s = \frac{1}{2\pi} \qquad f_s = 0 \tag{52}$$

Substitution of these values into Eqs. (29) and (31) gives, for $s = 0$

Distributed Load	Concentrated Load
$U_{01} = \dfrac{\Phi(\mu + \bar{\mu}\bar{k})}{4\pi\mu D_0}$	$U_{01} = \dfrac{\mu + \bar{\mu}\bar{k}}{4\pi\mu D_0}$
$U_{02} = \dfrac{\Phi(\mu k - \bar{\mu}\bar{k})\delta^2}{4\pi\mu D_0}$	$U_{02} = \dfrac{(\mu k - \bar{\mu}\bar{k})\delta^2}{4\pi\mu D_0}$
$\bar{U}_{01} = \dfrac{\Phi(k+1)}{4\pi D_0}$	$\bar{U}_{01} = \dfrac{k+1}{4\pi D_0}$
$\bar{U}_{02} = 0$	$\bar{U}_{02} = 0$

(53)

We will assign the value $V_{01} = 0$ to the arbitrary constant. Since the moment M is zero, Eqs. (27) give

$$V_{0j} = \bar{V}_{0j} = 0 \tag{54}$$

The constants required to evaluate the coefficients for $|s| = 1$ are obtained from Eqs. (34), (37), and (38). Eqs. (50) and

(52) show that for both the distributed and the concentrated normal load

$$g_{-s} = g_s \qquad f_s = 0 \tag{55}$$

The constants needed for $S = \pm 1$ are obtained using the quantities defined in Eq. (35). In particular, we find for the present case

$$\begin{aligned} Q_{-11} &= Q_{11} = -\frac{g_1}{2\mu\bar{\mu}(\lambda+1)(\bar{\lambda}+1)} \\ Q_{-12} &= Q_{12} = 0 \end{aligned} \tag{56}$$

We will define the arbitrary constants

$$U_{-13} = U_{13} = 0 \tag{57}$$

With these results, we observe from Eqs. (34) and (37) that

$$U_{-1j} = U_{1j} \qquad \bar{U}_{-1j} = \bar{U}_{1j} \tag{58}$$

and that these constants are all real. From Eqs. (18) and (21), therefore, we find

$$\begin{aligned} u_{-1} &= u_1 \qquad \sigma_{r,-1} = \sigma_{r1} \qquad \sigma_{\phi,-1} = \sigma_{\phi 1} \\ v_{-1} &= -v_1 \qquad \tau_{-1} = -\tau_1 \end{aligned} \tag{59}$$

for the barred as well as the unbarred values. We note also that the even coefficients are real and the odd coefficients are imaginary.

In a similar fashion, using Eqs. (40), (43), (44) to find the constants for $|s| \geq 2$, we find that

$$U_{-sj} = U_{s(5-j)} \qquad \bar{U}_{-sj} = \bar{U}_{s(5-j)} \tag{60}$$

In other words, the constants, associated with corresponding positive and negative indices and which multiply the same power of r, are equal. Therefore, Eqs. (14) and (22) show that

$$u_{-s} = u_s \qquad \sigma_{r,-s} = \sigma_{rs} \qquad \sigma_{\theta,-s} = \sigma_{\theta s}$$

$$v_{-s} = -v_s \qquad \tau_{-s} = -\tau_s \tag{61}$$

Here also, the even and odd coefficients are real and imaginary, respectively.

The inversion of the transformed solution requires only the cosine terms of Eq. (48) for the even coefficients and the sine terms for the odd. Thus

$$u = u_0 + 2\sum_{s=1}^{\infty} u_s \cos(s\theta)$$

$$v = 2i\sum_{s=1}^{\infty} v_s \sin(s\theta) \tag{62}$$

with corresponding formulas for the stresses.

The solution is now completely determined in series form. The necessary coefficients are given by Eqs. (14), (16), (18), (19), (20) and (21). The boundary values are given by Eq. (50) or (52)

depending on the load.

3. Uniform and Concentrated Shear Loads

The loading of a disc by a unit shear stress distributed over the arc $(-\Phi/2, \Phi/2)$ gives the boundary condition

$$\tau_{r\theta}(1,\theta) = f(\theta) = \begin{cases} 1 & (-\frac{\Phi}{2} < \theta < \frac{\Phi}{2}) \\ 0 & \text{elsewhere} \end{cases} \tag{63}$$

$$\sigma_r(1,\theta) = g(\theta) = 0$$

The boundary condition for the corresponding concentrated load is

$$\tau_{r\theta}(1,\theta) = f(\theta) = \begin{cases} \lim\limits_{\Phi \to 0} \frac{1}{2\sin(\Phi/2)} & (-\frac{\Phi}{2} < \theta < \frac{\Phi}{2}) \\ 0 & \text{elsewhere} \end{cases} \tag{64}$$

The transformed boundary conditions are

$$f_s = \begin{cases} \frac{\Phi}{2\pi} & (s = 0) \\ \frac{\sin(s\Phi/2)}{\pi s} & (s \neq 0) \end{cases} \tag{65}$$

and

$$f_s = \frac{1}{2\pi} \tag{66}$$

for the distributed and concentrated loads, respectively. In each case

$$g_s = 0 \tag{67}$$

We now can determine the coefficients needed to evaluate the series for displacements and stresses. If we assign to the arbitrary constant γ_{01} the value

$$V_{01} = - V_{02} \tag{68}$$

we can write from Eqs. (27)

$$V_{01} = f_0/2\mu$$

$$V_{02} = -f_0/2\mu$$

$$\overline{V}_{01} = f_0\left[\mu\delta^{-2} + \bar{\mu}(1-\delta^{-2})\right]/2\mu\bar{\mu} \tag{69}$$

$$\overline{V}_{02} = -f_0/2\bar{\mu}$$

From Eqs. (29) and (31) we find that the coefficients U_{0j} all vanish.

To find the coefficients for $s = \pm 1$, we observe first that the only non-vanishing right hand members of Eqs. (32) are in (b) and (g). Since these are odd in s and imaginary the constants U_{sj} and $\overline{U}_{sj}$ will have the same property, if we again define

$$U_{13} = U_{-13} = 0 \tag{70}$$

From Eqs. (18) and (21), therefore, we find that u_s, σ_{rs}, and $\sigma_{\theta s}$ are odd in s and imaginary while v_s and τ_s are even and real.

For $|s| \geqslant 2$, a symmetric or concentrated shear load is seen from Eqs. (40), (43), (44) and (47) to have the property

$$U_{-sj} = -U_{s(5-j)} \qquad U_{-sj} = -U_{s(5-j)} \tag{71}$$

Therefore u_s , σ_{rs} , and $\sigma_{\theta s}$ are odd in s and pure imaginary while v_s and τ_s are even and real. Combining the terms in the form shown by Eq. (48) we obtain

$$
\begin{aligned}
u &= 2i \sum_{s=1}^{\infty} u_s \sin(s\theta) \\
v &= v_0 + 2 \sum_{s=1}^{\infty} v_s \cos(s\theta)
\end{aligned}
\tag{72}
$$

where the coefficients have all been determined. Similar expressions hold for the stresses and the formulas hold for both the unbarred and barred variables.

C. Remarks on an Arbitrary Load

The examples just cited were chosen for illustrative purposes because of their simplicity. However, any load may be treated by the same technique. We can break an arbitrary load into four components, that is, the even and odd parts of the normal and shear components. The even part of the normal component and the odd part of the shear component lead to coefficients u_s , σ_{rs} , and $\sigma_{\theta s}$ which are even in s and real, and to v_s and τ_s which are odd and imaginary. The odd part of the normal component and the even part of the shear component lead to coefficients with the opposite character. Thus each function when expanded in the form of Eq. (48) is explicitly real. The solution for any load can then be written in series form, where the coefficients are given by the formulas developed above.

D. The Homogeneous Disc

The homogeneous disc may be treated as a limiting case of the

composite disc in several ways, such as letting the interface radius δ vanish or become unity, or by assigning the same set of values to the material constants in each region. Alternatively, the homogeneous disc may be treated as a boundary value problem in one region. The technique is straightforward by any of the methods. The solution has been obtained both by solving the problem directly and by equating the material constants. The agreement of the solutions provides a check on the calculations. Because of the relative simplicity of the series coefficients, we will write them explicitly. We prescribe the values

$$U_{-13} = U_{13} = 0 \qquad V_{01} = -V_{02} \tag{73}$$

to the rigid displacements. We obtain the values for the coefficients as follows:

For $s = 0$

$$\begin{aligned} u_0 &= g_0 r/2\mu k & v_0 &= -f_0(1-r^2)\,2/\mu r \\ \sigma_{r0} &= g_0 & \tau_0 &= f_0/r^2 \\ \sigma_{\theta 0} &= g_0 \end{aligned} \tag{74}$$

For $s = \pm 1$

$$\begin{aligned} u_s = &-\frac{g_s}{8\mu k(k+1)}\left[(k-2)r^2 - 2k(k+2)\log r\right] \\ &-\frac{isf_s}{8\mu k(k+1)}\left[(2k+1)(k-2)r^2 + 2k(k+2)\log r\right] \\ v_s = &-\frac{isg_s}{8\mu k(k+1)}\left[(3k+2)r^2 - 2k^2 - 2k(k+2)\log r\right] \\ &+\frac{f_s}{8\mu k(k+1)}\left[(3k+2)(2k+1)r^2 + 2k^2 + 2k(k+2)\log r\right] \end{aligned} \tag{75}$$

$$\sigma_{rs} = \frac{g_s}{2(k+1)}\left[r + (2k+1)r^{-1}\right] + \frac{(2k+1)isf_s}{2(k+1)}(r - r^{-1})$$

$$\sigma_{\theta s} = \frac{g_s}{2(k+1)}(3r - r^{-1}) + \frac{isf_s}{2(k+1)}\left[3(2k+1)r + r^{-1}\right] \qquad \begin{array}{c}(75)\\ \text{(Cont.)}\end{array}$$

$$\tau_s = -\frac{isg_s}{2(k+1)}(r - r^{-1}) + \frac{f_s}{2(k+1)}\left[(2k+1)r + r^{-1}\right]$$

For $|s| \geq 2$ it will be convenient to work only with positive values of s. We combine the positive and negative values of the index by defining

$$\begin{aligned} G_s &= g_s + \gamma g_{-s} \qquad & F_s &= f_s + \gamma f_{-s} \\ \tilde{G}_s &= g_s - \gamma g_{-s} \qquad & \tilde{F}_s &= f_s - \gamma f_{-s} \end{aligned} \qquad (76)$$

where the symbol γ is assigned the value plus or minus one, as appropriate. The coefficients for the series are then

$$\begin{aligned} u_s + \gamma u_{-s} &= -\frac{sk-2}{4\mu k(s+1)}(G_s + i\tilde{F}_s)r^{s+1} \\ &\quad + \frac{1}{4\mu(s-1)}\left[sG_s + i(s-2)\tilde{F}_s\right]r^{s-1} \qquad (77) \\ v_s + \gamma v_{-s} &= -\frac{sk+2k+2}{4\mu k(s+1)}(i\tilde{G}_s - F_s)r^{s+1} \\ &\quad + \frac{1}{4\mu(s-1)}\left[is\tilde{G}_s - (s-2)F_s\right]r^{s-1} \end{aligned}$$

$$\sigma_{rs} + \gamma\sigma_{r,-s} = \frac{1}{2}\Big\{-(s-2)(G_s + i\tilde{F}_s)r^s$$

$$+ \left[sG_s + i(s-2)\tilde{F}_s\right] r^{s-2}\Big\}$$

$$\sigma_{\theta s} + \gamma\sigma_{\theta,-s} = \frac{1}{2}\Big\{(s+2)(G_s + i\tilde{F}_s)r^s \qquad (77)$$

$$-\left[sG_s + i(s-2)\tilde{F}_s\right]r^{s-2}\Big\}$$

$$\tau_s + \gamma\tau_{-s} = \frac{1}{2}\Big\{-s(i\tilde{G}_s - F_s)r^s$$

$$+\left[is\tilde{G}_s - (s-2)F_s\right] r^{s-2}\Big\}$$

We have now exhibited explicitly the coefficients required for the solution in series form for any loading on a homogeneous disc.

For the four specific loads described above, we can carry the solution a bit further. For the unit normal load, distributed over an arc Φ, the explicit solution is

$$u = \frac{1}{4\pi\mu}\Big\{\frac{\Phi r}{k}$$

$$-\left[(k-2)r^2 + 2k(k+2)\log r\right]\frac{\sin(\Phi/2)\cos\theta}{k(k+1)} \qquad (78)$$

$$-2\sum_{s=2}^{\infty}\left[\frac{ks-2}{k\,s(s+1)}r^{s+1} - \frac{1}{s-1}r^{s-1}\right]\sin(s\Phi/2)\cos(s\theta)\Big\}$$

$$v = \frac{1}{4\pi\mu}\left\{\left[(3k+2)r^2 - 2k^2 - 2k(k+2)\log r\right]\frac{\sin(\Phi/2)\sin\theta}{k(k+1)}\right.$$

$$\left. + 2\sum_{s=2}^{\infty}\left[\frac{ks+2k+2}{ks(s+1)}r^{s+1} - \frac{1}{s-1}r^{s-1}\right]\sin(s\Phi/2)\sin(s\theta)\right\}$$

$$\sigma_r = \frac{1}{\pi}\left\{\frac{\Phi}{2} + \left[r + (2k+1)r^{-1}\right]\frac{\sin(\Phi/2)\cos\theta}{k+1}\right.$$

$$\left. - \sum_{s=2}^{\infty}\left(\frac{s-2}{s}r^s - r^{s-2}\right)\sin(s\Phi/2)\cos(s\theta)\right\}$$

(78)
(Cont.)

$$\sigma_\theta = \frac{1}{\pi}\left\{\frac{\Phi}{2} + (3r - r^{-1})\frac{\sin(\Phi/2)\cos\theta}{k+1}\right.$$

$$\left. + \sum_{s=2}^{\infty}\left(\frac{s+2}{s}r^s - r^{s-2}\right)\sin(s\Phi/2)\cos(s\theta)\right.$$

$$\tau_{r\theta} = \frac{1}{\pi}\left\{(r - r^{-1})\frac{\sin(\Phi/2)\sin\theta}{k+1}\right.$$

$$\left. + \sum_{s=2}^{\infty}(r^s - r^{s-2})\sin(s\Phi/2)\sin(s\theta)\right.$$

Let us define

$$\alpha = \Phi/2 + \theta \qquad \beta = \Phi/2 - \theta \tag{79}$$

Combining the terms of Eq. (77), we obtain series expressions of the form

$$\begin{aligned} C(r,\omega) &= \sum_{s=1}^{\infty} \frac{r^s \cos(s\omega)}{s} \\ S(r,\omega) &= \sum_{s=1}^{\infty} \frac{r^s \sin(s\omega)}{s} \\ K(r,\omega) &= \sum_{s=1}^{\infty} r^s \cos(s\omega) \\ Z(r,\omega) &= \sum_{s=1}^{\infty} r^s \sin(s\omega) \end{aligned} \tag{80}$$

These series are all convergent inside the unit circle, to the formulas given by Jolley[17].

$$\begin{aligned} C(r,\omega) &= -(1/2) \log H(r,\omega) \\ S(r,\omega) &= \tan^{-1} \frac{r \sin\omega}{1 - r\cos\omega} \\ K(r,\omega) &= \frac{r(\cos\omega - r)}{H(r,\omega)} \\ Z(r,\omega) &= \frac{r\sin\omega}{H(r,\omega)} \end{aligned} \tag{81}$$

where

$$H(r,\omega) = 1 + r^2 - 2r\cos\omega$$

In the same fashion we can examine each of the other loadings. For the concentrated loads we need two more series expressions.

$$Q(r,\omega) = \sum_{s=1}^{\infty} s r^s \cos(s\omega) = \frac{r[(1+r^2)\cos\omega - 2r]}{H^2(r,\omega)}$$

(82)

$$P(r,\omega) = \sum_{s=1}^{\infty} s r^s \sin(s\omega) = \frac{r(1-r^2)\sin\omega}{H^2(r,\omega)}$$

Eqs. (81) and (82) hold in the interior of the unit circle. By analytic continuation we can take them to hold at those points on the unit circle where the formulas exist. It is precisely at the discontinuities of the loads that the expressions become singular.

On the following pages we tabulate the closed form solutions for the four problems examined. The formulas for concentrated load agree with those derived from the stress functions given by Mindlin[25] and Ng[28]. The concentrated normal load disagrees with that derived from Michell[22], but apparently a term was left out of his formula.

DISTRIBUTED UNIT NORMAL STRESS

$$
\begin{aligned}
u = \frac{1}{4\pi\mu}\Big\{ & \frac{\Phi r}{k} + \left[(k-2)r^2 + 2(k+2)\log r\right]\frac{\sin\alpha + \sin\beta}{2(k+1)} \\
& + \frac{2}{k}\big[(k+1)C(r,\alpha)\sin\alpha + (k+1)C(r,\beta)\sin\beta \\
& - S(r,\alpha)(\cos\alpha - r) - S(r,\beta)(\cos\beta - r)\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
v = \frac{1}{4\pi\mu}\Big\{ & \left[(3k+2)r^2 + 2(k+2)\log r + 2k\right]\frac{\cos\alpha - \cos\beta}{2(k+1)} \\
& + \frac{2}{k}\big[(k+1)C(r,\alpha)(\cos\alpha - r) - (k+1)C(r,\beta)(\cos\beta - r) \\
& + S(r,\alpha)\sin\alpha - S(r,\beta)\sin\beta\big]\Big\}
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
\sigma_r = \frac{1}{2\pi}\Big\{ & \Phi - (kr^2 - 2k - 1)\frac{\sin\alpha + \sin\beta}{(k+1)r} \\
& + 2\left[S(r,\alpha) + S(r,\beta)\right] + 2r^{-1}\left[K(r,\alpha)\sin\alpha + K(r,\beta)\sin\beta\right]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_\theta = \frac{1}{2\pi}\Big\{ & \Phi - (3kr^2 + 1)\frac{\sin\alpha + \sin\beta}{(k+1)r} \\
& + 2\left[S(r,\alpha) + S(r,\beta)\right] - 2r^{-1}\left[K(r,\alpha)\sin\alpha + K(r,\beta)\sin\beta\right]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\tau_{r\theta} = \frac{1}{2\pi}\Big\{ & (kr^2 + 1)\frac{\cos\alpha - \cos\beta}{(k+1)r} \\
& - 2r^{-1}\left[Z(r,\alpha)\sin\alpha - Z(r,\beta)\sin\beta\right]\Big\}
\end{aligned}
$$

DISTRIBUTED UNIT SHEAR STRESS

$$
\begin{aligned}
u = \frac{1}{4\pi\mu} \Big\{ & -\left[(k-2)r^2 + 4(k+1) + 2(k+2)\log r\right] \frac{\cos\alpha - \cos\beta}{2(k+1)} \\
& + \frac{2}{k}\Big[-(k+1)\left[C(r,\alpha)\cos\alpha - C(r,\beta)\cos\beta\right] \\
& + (r + k r^{-1})\left[C(r,\alpha) - C(r,\beta)\right] - S(r,\alpha)\sin\alpha + S(r,\beta)\sin\beta \Big] \Big\} \\
v = \frac{1}{4\pi\mu} \Big\{ & -\frac{\Phi}{r} + \left[(3k+2)(r^2+2) + 2(k+2)\log r\right] \frac{\sin\alpha + \sin\beta}{2(k+1)} \\
& + \frac{2}{k}\Big[\left[(k+1)r - k r^{-1}\right]\left[S(r,\alpha) + S(r,\beta)\right] - S(r,\alpha)\cos\alpha \\
& - S(r,\beta)\cos\beta + (k+1)\left[C(r,\alpha)\sin\alpha + C(r,\beta)\sin\beta\right] \Big] \Big\} \\
\sigma_r = \frac{1}{2\pi} \Big\{ & (k r^2 - 2k - 1)\frac{\cos\alpha - \cos\beta}{(k+1)r} + 2(1 - r^{-2})\left[C(r,\alpha) - C(r,\beta)\right] \\
& + (k+3)r^{-1}\left[K(r,\beta)\cos\beta - K(r,\alpha)\cos\alpha\right] \qquad (84) \\
& - (k+1)r^{-1}\left[Z(r,\alpha)\sin\alpha - Z(r,\beta)\sin\beta\right] + (k+1+2r^{-2})\left[K(r,\alpha) - K(r,\beta)\right] \Big\} \\
\sigma_\theta = \frac{1}{2\pi} \Big\{ & (3k r^2 + 1)\frac{\cos\alpha - \cos\beta}{(k+1)r} + 2(1 + r^{-2})\left[C(r,\alpha) - C(r,\beta)\right] \\
& + (k+3-2r^{-2})\left[K(r,\alpha) - K(r,\beta)\right] - (k+1)r^{-1}\left[K(r,\alpha)\cos\alpha - K(r,\beta)\cos\beta\right] \\
& - (k+3)r^{-1}\left[Z(r,\alpha)\sin\alpha - Z(r,\beta)\sin\beta\right] \Big\} \\
\tau_{r\theta} = \frac{1}{2\pi} \Big\{ & \Phi r^{-2} + (k r^2 + 1)\frac{\sin\alpha + \sin\beta}{(k+1)r} \\
& + r^{-1}\left[Z(r,\alpha)\cos\alpha + Z(r,\beta)\cos\beta\right] + (1 - 2r^{-2})\left[Z(r,\alpha) + Z(r,\beta)\right] \\
& + r^{-1}\left[K(r,\alpha)\sin\alpha + K(r,\beta)\sin\beta\right] + 2r^{-2}\left[S(r,\alpha) + S(r,\beta)\right] \Big\}
\end{aligned}
$$

CONCENTRATED UNIT NORMAL FORCE

$$u = \frac{1}{4\pi\mu}\Big\{-\frac{k+1}{k}r + \Big[(k-2)r^2 - 2(k+1) + 2(k+2)\log r\Big]\frac{\cos\theta}{2(k+1)}$$

$$+\frac{2}{k}\Big[(k+1)C(r,\theta)\cos\theta + S(r,\theta)\sin\theta\Big] - (r - r^{-1})K(r,\theta)\Big\}$$

$$v = \frac{1}{4\pi\mu}\Big\{-\Big[(3k+2)r^2 - 2 + 2(k+2)\log r\Big]\frac{\sin\theta}{2(k+1)}$$

$$+\frac{2}{k}\Big[S(r,\theta)\cos\theta - (k+1)C(r,\theta)\sin\theta\Big] + (r - r^{-1})Z(r,\theta)\Big\}$$

$$\sigma_r = \frac{1}{2\pi}\Big\{1 - (kr^2 - 2k - 1)\frac{\cos\theta}{(k+1)r} \tag{85}$$

$$+ 2r^{-1}\Big[(r + \cos\theta)K(r,\theta) - P(r,\theta)\sin\theta\Big]\Big\}$$

$$\sigma_\theta = \frac{1}{2\pi}\Big\{-1 - (3kr^2 + 1)\frac{\cos\theta}{(k+1)r}$$

$$+ 2r^{-1}\Big[Z(r,\theta) + P(r,\theta)\Big]\sin\theta\Big\}$$

$$\tau_{r\theta} = \frac{1}{2\pi}\Big\{-(kr^2 + 1)\frac{\sin\theta}{(k+1)r}$$

$$- 2r^{-1}\Big[Q(r,\theta)\sin\theta + Z(r,\theta)\cos\theta\Big]\Big\}$$

CONCENTRATED UNIT SHEAR STRESS

$$\begin{aligned} u = \frac{1}{4\pi\mu} \Big\{ & \Big[(k-2)r^2 + 2(k+1) + 2(k+2)\log r\Big] \frac{\sin\theta}{2(k+1)} \\ & + \frac{2}{k}\Big[(k+1)C(r,\theta)\sin\theta - S(r,\theta)\cos\theta\Big] \\ & + (r - r^{-1})Z(r,\theta) \Big\} \end{aligned}$$

$$\begin{aligned} v = \frac{1}{4\pi\mu} \Big\{ & \Big[(3k+2)r^2 + 2(2k+1) + 2(k+2)\log r\Big] \frac{\cos\theta}{2(k+1)} \\ & + \frac{2}{k}\Big[(k+1)C(r,\theta)\cos\theta - S(r,\theta)\sin\theta - r\Big] \\ & + (r - r^{-1})K(r,\theta) - (r + r^{-1}) \Big\} \end{aligned}$$

$$\begin{aligned} \sigma_r = \frac{1}{2\pi} \Big\{ & (2k + 1 - k r^2)\frac{\sin\theta}{(k+1)r} \\ & + (k + r^{-2})Z(r,\theta) + (1 - r^{-2})P(r,\theta) \\ & + (k+3)r^{-1}K(r,\theta)\sin\theta - (k+1)r^{-1}Z(r,\theta)\cos\theta \Big\} \end{aligned} \tag{86}$$

$$\begin{aligned} \sigma_\theta = \frac{1}{2\pi} \Big\{ & -(1 + 3k r^2)\frac{\sin\theta}{(k+1)r} \\ & + (k - r^{-2})Z(r,\theta) - (1 - r^{-2})P(r,\theta) \\ & + (k+1)r^{-1}K(r,\theta)\sin\theta - (k+3)r^{-1}Z(r,\theta)\cos\theta \Big\} \end{aligned}$$

$$\begin{aligned} \tau_{r\theta} = \frac{1}{2\pi} \Big\{ & (1 + k r^2)\frac{\cos\theta}{(k+1)r} + r^{-2} + r^{-2}K(r,\theta) \\ & + (1 - r^{-2})Q(r,\theta) - r^{-1}Z(r,\theta)\sin\theta + r^{-1}K(r,\theta)\cos\theta \Big\} \end{aligned}$$

PART IV

NUMERICAL RESULTS

In a formal sense, the solution is now completed and all required numerical values can be obtained from the summation of the series. In practice, however, this summation can not be done for certain configurations. In particular, if the ring thickness, $h = 1 - \delta$, is small, convergence is so slow as to be meaningless. We can see this more clearly if we look at one of the series expressions in detail.

$$
\begin{aligned}
u = u_0 + u_1 + \sum_{s=2}^{\infty} \Big\{ & \left[a_1 + (a_2 s + a_3)\delta^{2s} \right] r^{s+1} \\
& + \left[b_1 s + (b_2 + b_3 s + b_4 s^2)\delta^{2s} \right] r^{s-1} \\
& + \left[(c_1 + c_2 s + c_3 s^2)\delta^{2s} + c_4 s \delta^{4s} \right] r^{-s-1} \\
& + \left[(d_1 + d_2 s)\delta^{2s} + d_3 \delta^{4s} \right] r^{-s+1} \Big\} \frac{\sin(s\Phi/2)\cos(s\theta)}{D_s}
\end{aligned}
\tag{87}
$$

where

$$
D_s = e_1 + (e_2 + e_3 s^2)\delta^{2s} + e_4 \delta^{4s} \tag{88}
$$

and where a_j, b_j, c_j, d_j, and e_j are constants dependent on material properties and independent of r and s. This series represents the formula for radial displacement in the annulus of a composite disc of unit outer radius loaded by normal stress uniformly distributed over the arc $(-\Phi/2, \Phi/2)$. The formulas for other quantities are similar.

Terms of the type $C_s s^2 \delta^{2s} r^{-s-1}$, for example, increase for a substantial number of terms before starting to decrease. For a small value of Φ the factor $\sin(s\Phi/2)$ also increases with s for many terms. As a result of these conditions convergence is very poor in many cases. For example, a case in which λ and Φ each had the value .01 was run for five thousand terms with no sign of convergence.

A scheme was developed for improving the convergence of the series by establishing as a first approximation the solution to the case for the homogeneous disc. The approximation is then corrected by adding to it the difference between the corresponding terms of the composite and the homogeneous disc solutions. Since the terms of the two series approach one another asymptotically this offers a chance of improvement. Two forms of this scheme were used. In one, the material properties used for the homogeneous disc were those of the core region of the composite disc; for the second the properties used were those of the ring. In the homogeneous disc, the only elastic property affecting the stresses is Poisson's ratio. This dependence shows up in the series in only the term for $s = \pm 1$. Therefore, regardless of the form used for the first approximation, the speed of convergence for the stress series is the same. For the displacements, however, the choice of first approximation does make a difference. It was found that about half as many terms are required if the ring properties are used. The bulk of the computations, however, had been performed using the properties of the core. These computations were spot checked using the more rapidly convergent technique.

Numerical values have been obtained for selected points on the outer surface and along the interface for the case of unit normal load acting on a composite disc in plane stress. Two configurations of material constants have been used, corresponding respectively to a layer of gold or alumina on a base of steel. The properties of these materials are shown below in Table I.

Table I. Material Properties

Material	μ (psi)	ν	k	$\mu/\bar{\mu}$
Steel	$1.153(10)^7$	0.300	1.86	1.000
Gold	$4.014(10)^6$	0.420	2.45	0.3481
Alumina	$2.086(10)^7$	0.246	1.34	2.418

Values are calculated for a dimensionless configuration of unit radius, unit shear modulus in the base region, and unit normal stress. We can convert results to a system with dimensionalized quantities. Let $\hat{u}(\hat{r},\theta)$ and $\hat{\sigma}(\hat{r},\theta)$ be the displacements and stresses in a disc of radius ρ, with a shear modulus $\hat{\mu}$ in the core. We can convert the dimensionless results to values for a corresponding dimensionalized disc by formulas such as

$$\hat{u}(\hat{r},\theta) = \frac{\rho}{\hat{\mu}}\, u(\hat{r}/\rho,\theta)$$

$$\hat{\sigma}(\hat{r},\theta) = \sigma(\hat{r}/\rho,\theta) \tag{29}$$

Table II lists some results for the disc whose layer thickness λ and load width Φ both are equal to 0.01. Values in the vicinity of the load, both on the surface and at the interface, are

given. A jump discontinuity in the hoop stress at the interface is indicated by listing as σ_θ^A the value in the ring and as σ_θ^B the value in the core. A list is shown for both a gold (soft) and alumina (hard) layer on a steel core. Figure 2 shows graphically these and some other results.

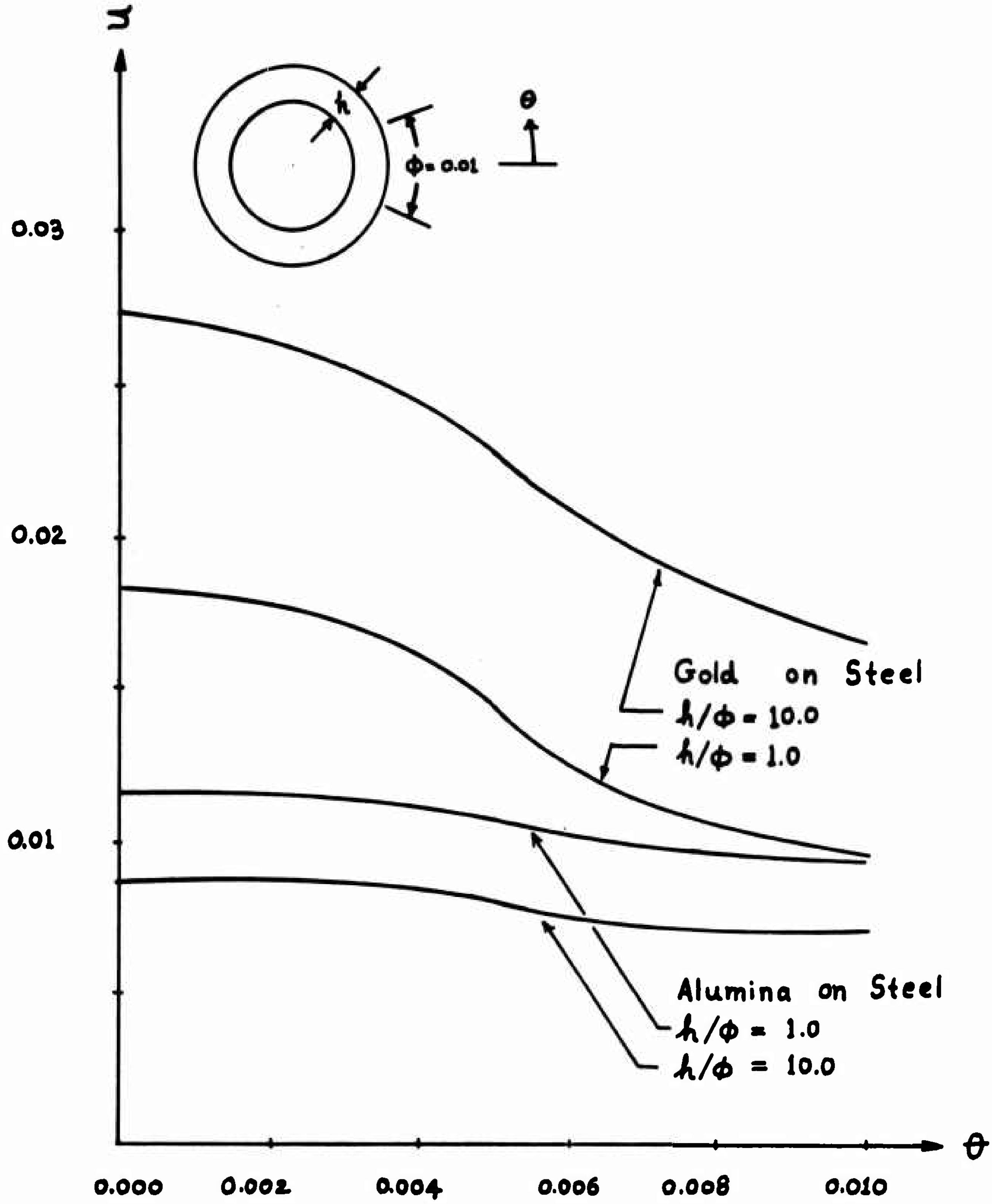

Figure 2. Behavior of a Composite Disc
Part A. Radial Displacement of the Surface

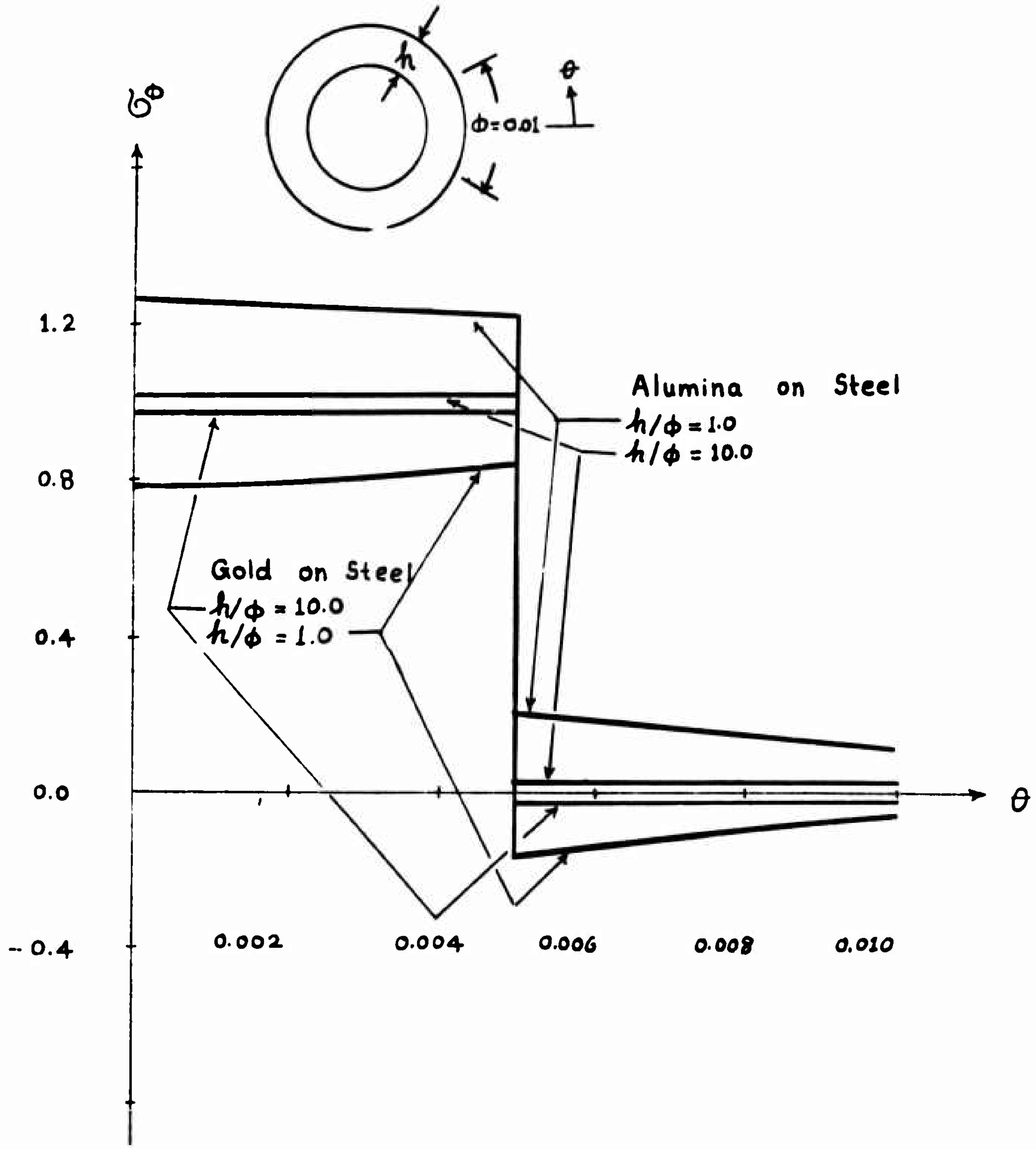

Figure 2. Behavior of a Composite Disc
Part B. σ_θ at the Surface

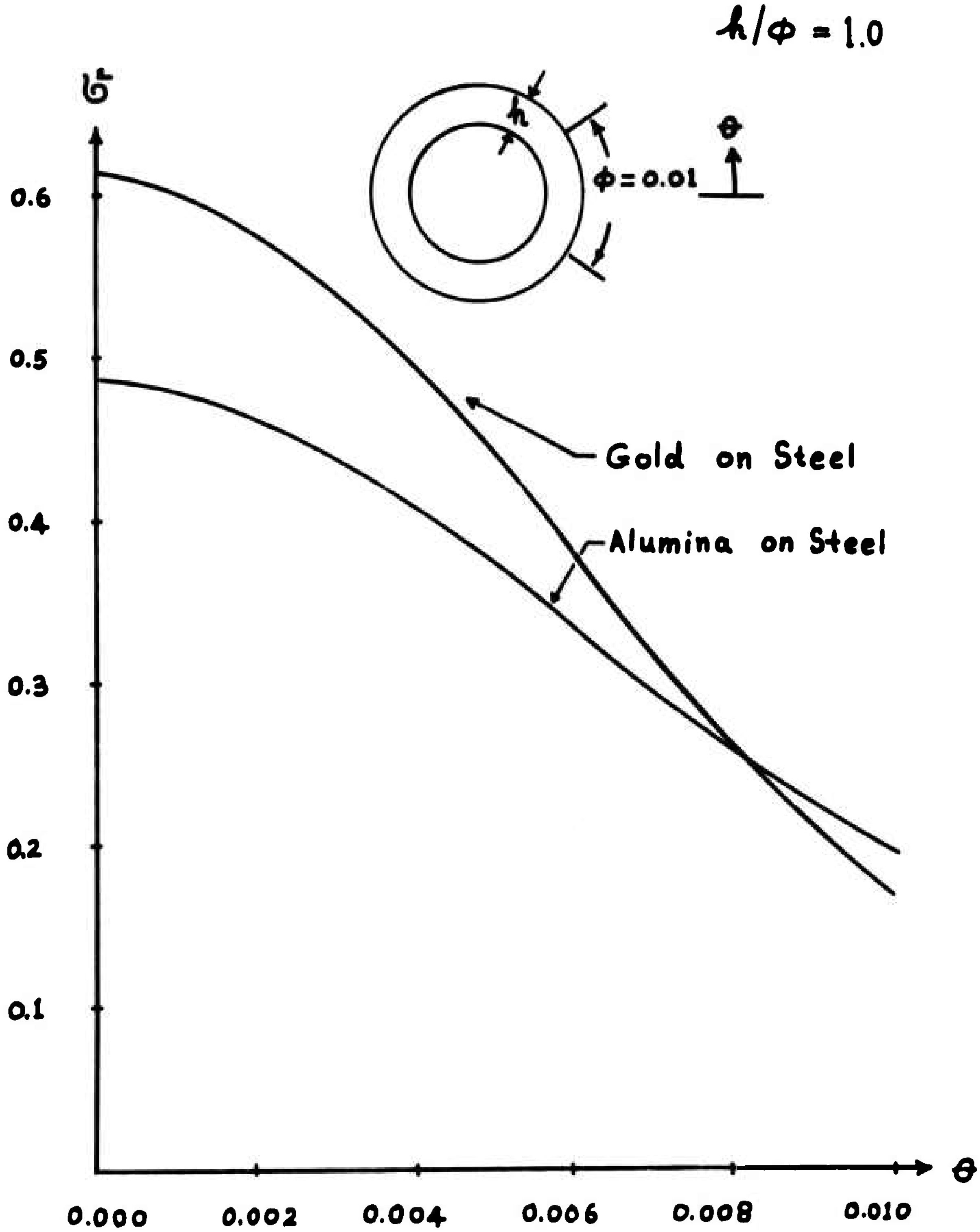

Figure 2. Behavior of a Composite Disc

Part C. σ_r at the Interface.

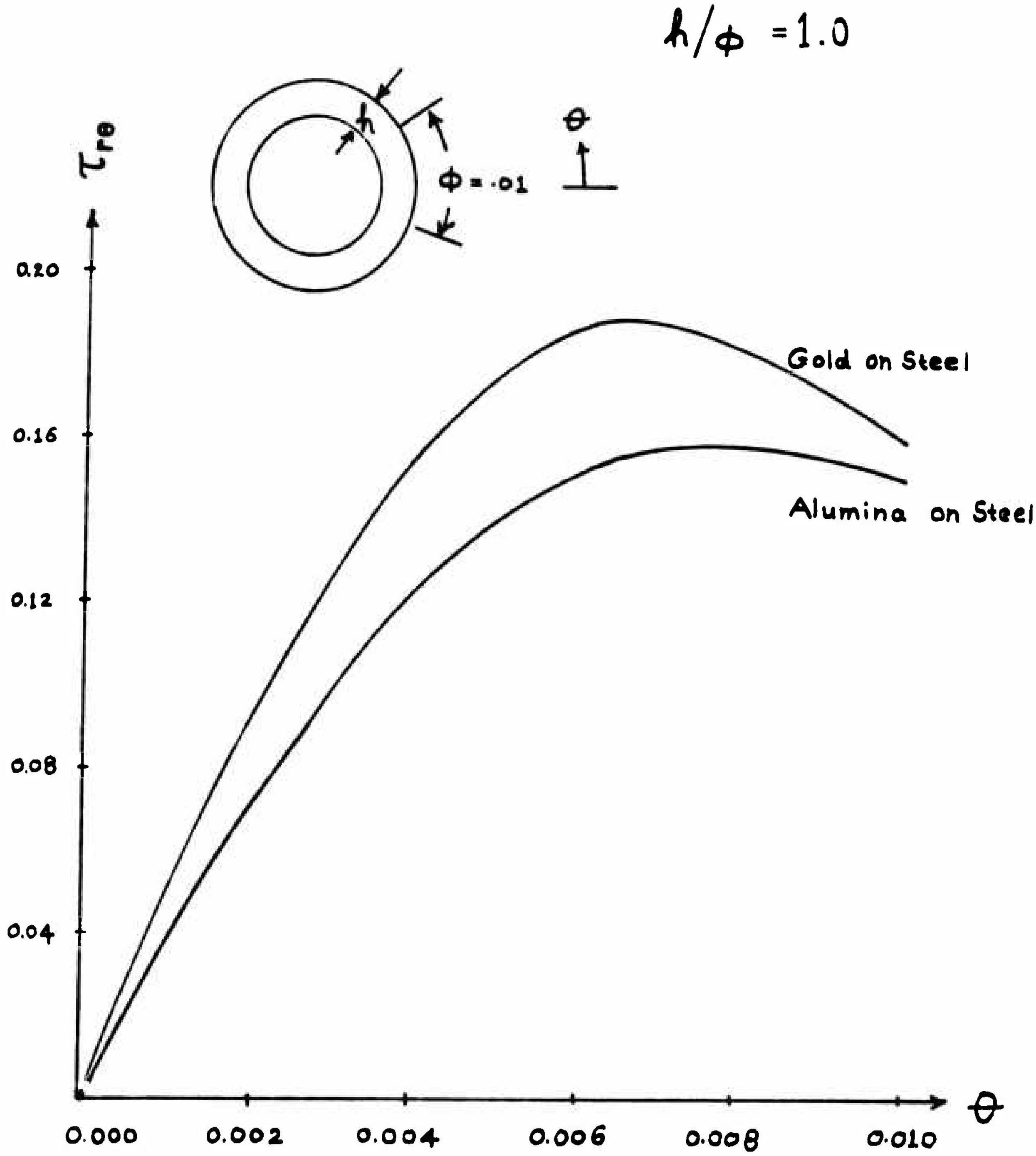

Figure 2. Behavior of a Composite Disc
Part D. $\tau_{r\theta}$ at the Interface.

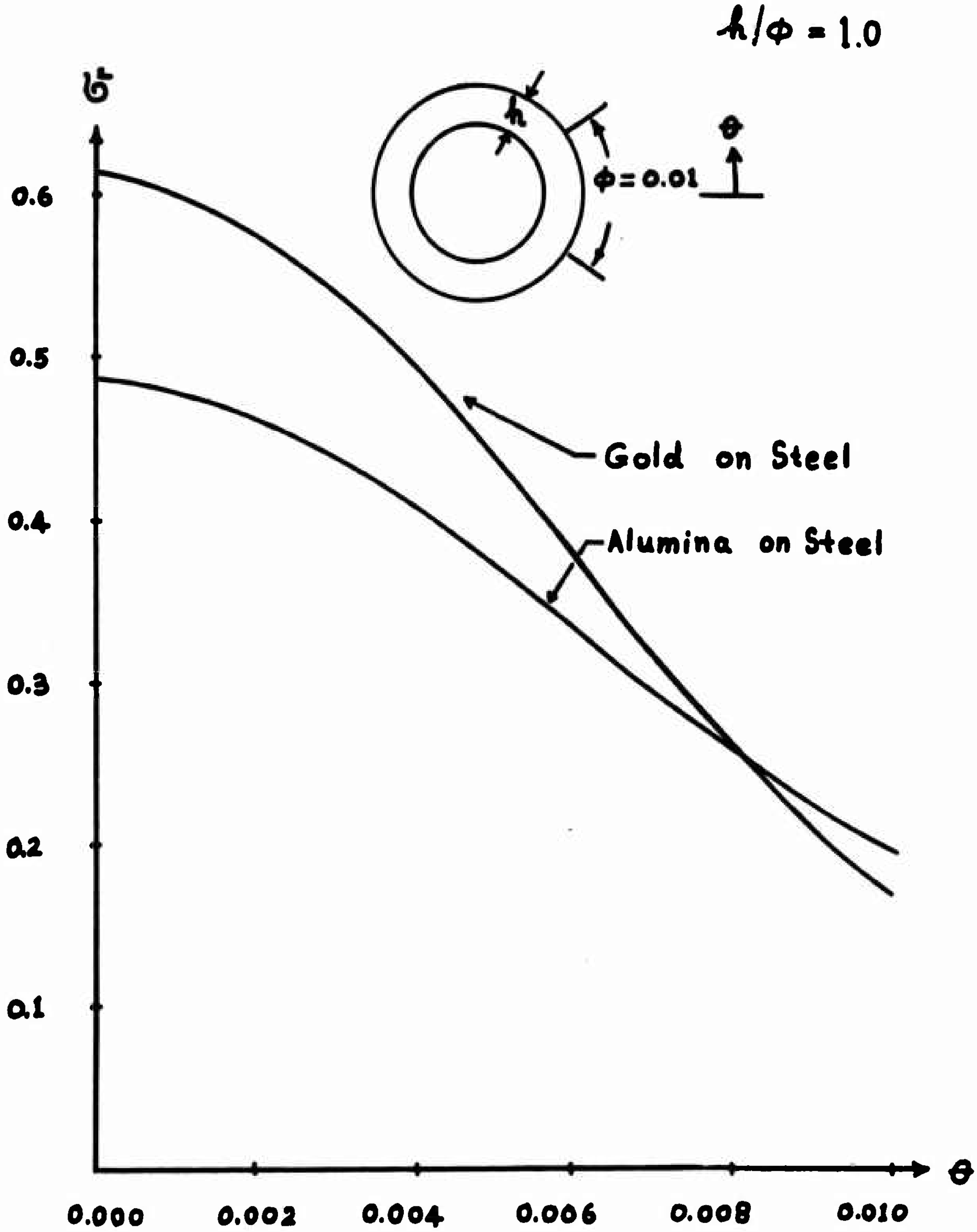

Figure 2. Behavior of a Composite Disc

Part C. σ_r at the Interface.

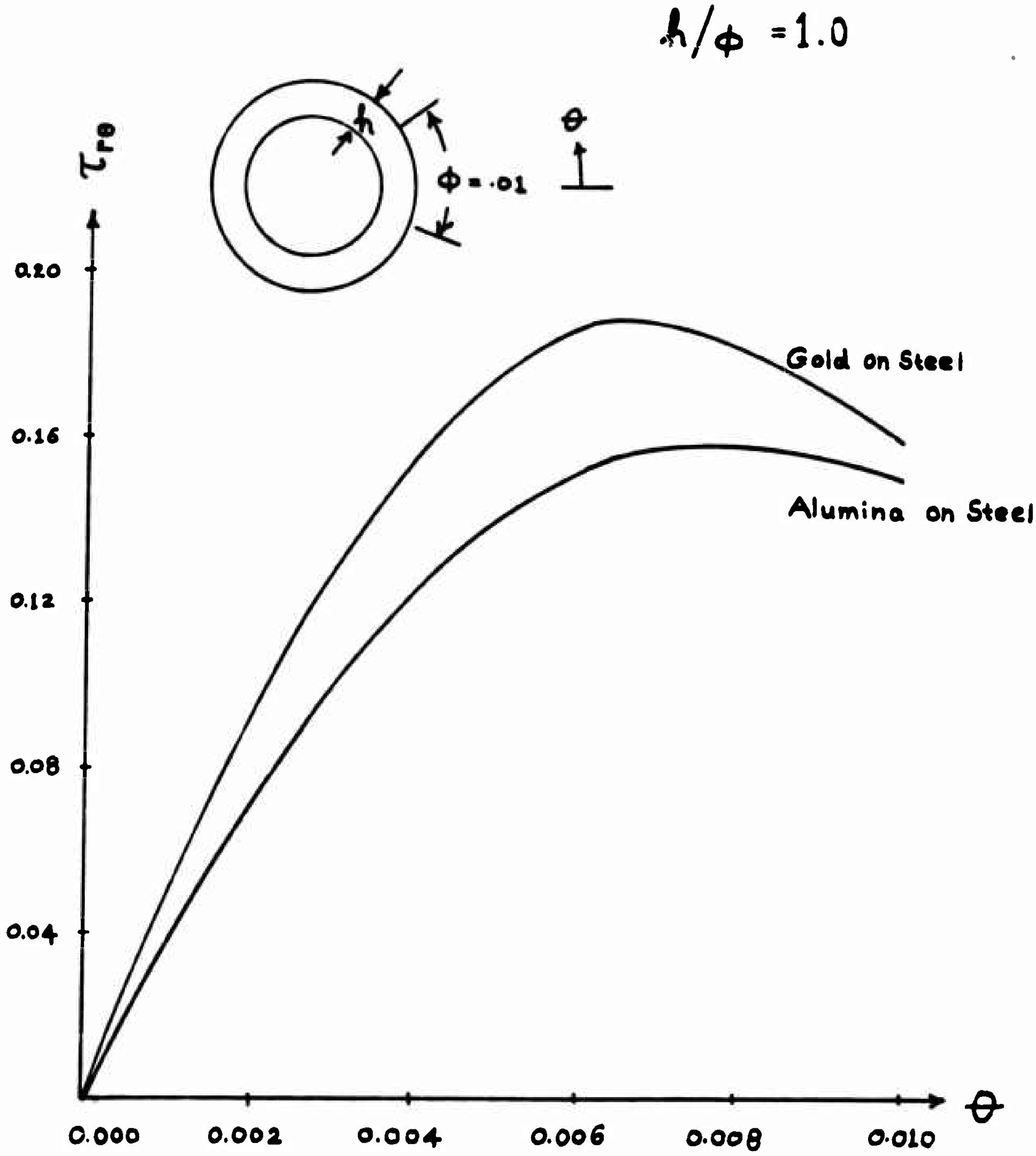

Figure 2. Behavior of a Composite Disc
Part D. $\tau_{r\theta}$ at the Interface.

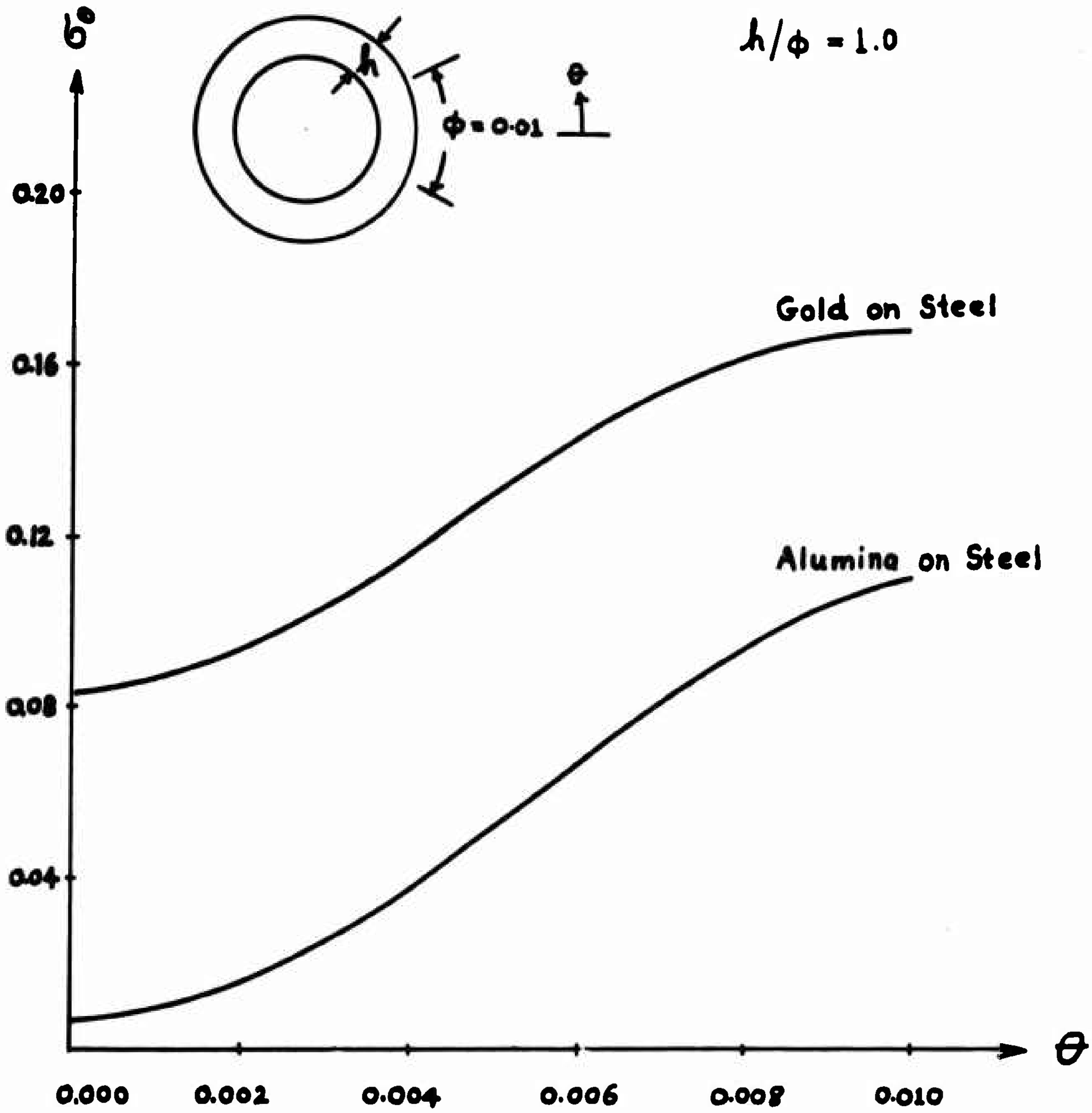

Figure 2. Behavior of a Composite Disc
Part E. σ_ϕ at the Interface in the Base.

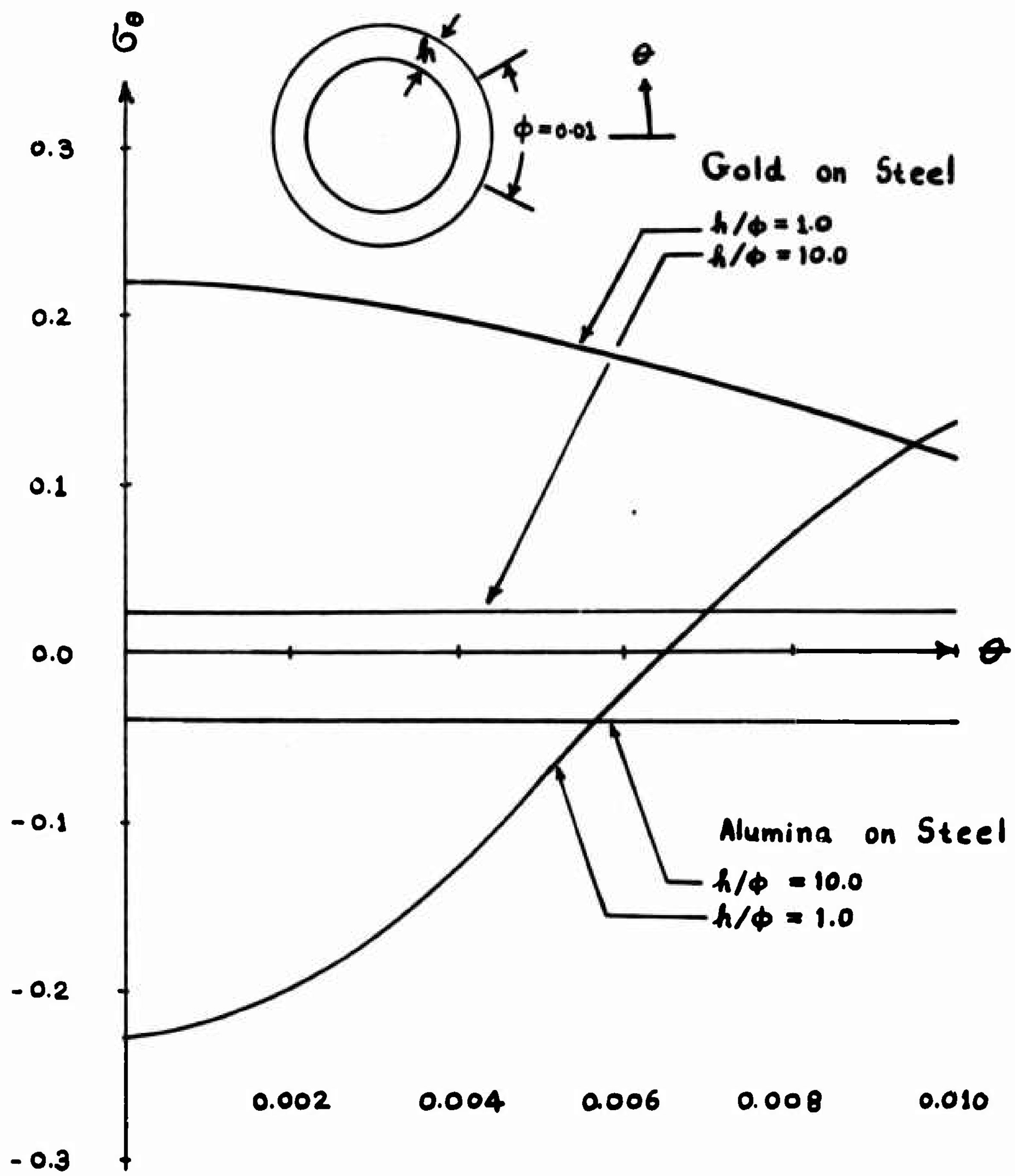

Figure 2. Behavior of a Composite Disc
Part F. σ_θ at the Interface in the Ring.

Table II

DISPLACEMENTS AND STRESSES*

Part A. Gold Layer on Steel

100 × θ	Surface 100 u	Surface σ_θ	Interface 100 u	Interface σ_r	Interface $\tau_{r\theta}$	Interface σ_θ^A	Interface σ_θ^B
0.00	1.833	0.797	1.096	0.612	0.000	0.219	0.083
0.05	1.831	0.798	1.096	0.610	-0.023	0.218	0.084
0.10	1.823	0.799	1.094	0.603	-0.045	0.217	0.086
0.15	1.809	0.802	1.091	0.594	-0.066	0.215	0.089
0.20	1.790	0.805	1.087	0.580	-0.087	0.212	0.092
0.25	1.763	0.810	1.081	0.563	-0.106	0.209	0.097
0.30	1.730	0.815	1.075	0.543	-0.123	0.205	0.103
0.35	1.686	0.821	1.068	0.519	-0.139	0.200	0.109
0.40	1.631	0.828	1.060	0.494	-0.153	0.195	0.116
0.45	1.560	0.835	1.051	0.467	-0.164	0.189	0.123
0.50	1.441	-	1.042	0.438	-0.173	0.183	0.130
0.55	1.321	-0.148	1.032	0.408	-0.180	0.177	0.137
0.60	1.247	-0.140	1.021	0.378	-0.184	0.170	0.143
0.65	1.188	-0.131	1.011	0.349	-0.187	0.163	0.149
0.70	1.138	-0.122	1.000	0.319	-0.187	0.156	0.155
0.75	1.097	-0.113	0.988	0.291	-0.185	0.150	0.159
0.80	1.060	-0.103	0.977	0.264	-0.182	0.143	0.163
0.85	1.029	-0.094	0.965	0.238	-0.177	0.136	0.166
0.90	1.001	-0.086	0.954	0.214	-0.172	0.129	0.168
0.95	0.976	-0.077	0.943	0.191	-0.165	0.123	0.169
1.00	0.954	-0.069	0.932	0.170	-0.158	0.116	0.169
1.25	0.871	-0.033	0.880	0.093	-0.118	0.089	0.158
1.50	0.817	-0.009	0.834	0.050	-0.084	0.068	0.138
1.75	0.778	0.004	0.795	0.028	-0.058	0.053	0.116
2.00	0.747	0.011	0.762	0.017	-0.041	0.042	0.097
2.25	0.721	0.013	0.732	0.011	-0.030	0.034	0.081
2.50	0.697	0.012	0.706	0.008	-0.022	0.028	0.068
2.75	0.676	0.010	0.683	0.006	-0.017	0.024	0.058
3.00	0.656	0.008	0.661	0.005	-0.013	0.020	0.050
3.25	0.637	0.006	0.642	0.004	-0.011	0.018	0.043
3.50	0.620	0.005	0.623	0.003	-0.009	0.015	0.038
3.75	0.603	0.003	0.606	0.003	-0.007	0.013	0.033
4.00	0.588	0.002	0.591	0.002	-0.006	0.012	0.029
4.25	0.574	0.001	0.576	0.002	-0.005	0.010	0.026
4.50	0.560	0.001	0.562	0.001	-0.004	0.009	0.023
4.75	0.547	0.000	0.549	0.001	-0.003	0.008	0.021
5.00	0.535	0.000	0.536	0.001	-0.003	0.007	0.018

*in a composite disc of unit radius loaded by a unit tensile radial stress over the arc (-0.005, 0.005) and by a concentrated reaction at the center. Layer thickness is 0.01.

Table II

DISPLACEMENTS AND STRESSES*

Part B. Alumina Layer on Steel

100 × θ	Surface		Interface				
	100 u	σ_θ	100 u	σ_r	$\tau_{r\theta}$	σ_θ^A	σ_θ^B
0.00	1.151	1.271	1.020	0.485	0.000	-0.227	0.006
0.05	1.150	1.271	1.020	0.483	-0.017	-0.225	0.006
0.10	1.148	1.269	1.018	0.480	-0.034	-0.220	0.008
0.15	1.145	1.266	1.017	0.474	-0.051	-0.211	0.011
0.20	1.139	1.262	1.014	0.465	-0.068	-0.199	0.015
0.25	1.133	1.257	1.010	0.455	-0.082	-0.184	0.019
0.30	1.124	1.250	1.006	0.442	-0.096	-0.166	0.025
0.35	1.114	1.243	1.002	0.428	-0.109	-0.146	0.031
0.40	1.101	1.236	0.996	0.412	-0.120	-0.124	0.038
0.45	1.084	1.227	0.990	0.395	-0.130	-0.100	0.045
0.50	1.059	-	0.984	0.377	-0.138	-0.075	0.052
0.55	1.033	0.208	0.977	0.358	-0.145	-0.050	0.060
0.60	1.015	0.198	0.970	0.339	-0.150	-0.024	0.067
0.65	0.999	0.187	0.963	0.320	-0.154	0.001	0.075
0.70	0.985	0.176	0.955	0.301	-0.157	0.025	0.082
0.75	0.972	0.165	0.947	0.282	-0.158	0.048	0.088
0.80	0.960	0.154	0.939	0.263	-0.158	0.070	0.094
0.85	0.949	0.142	0.931	0.246	-0.157	0.090	0.099
0.90	0.938	0.131	0.922	0.229	-0.155	0.108	0.104
0.95	0.927	0.121	0.914	0.213	-0.152	0.124	0.108
1.00	0.917	0.110	0.906	0.198	-0.149	0.139	0.111
1.25	0.871	0.063	0.865	0.135	-0.127	0.186	0.119
1.50	0.830	0.028	0.827	0.093	-0.104	0.201	0.116
1.75	0.794	0.005	0.792	0.064	-0.084	0.197	0.107
2.00	0.761	-0.010	0.761	0.044	-0.068	0.185	0.097
2.25	0.731	-0.017	0.732	0.031	-0.055	0.169	0.087
2.50	0.705	-0.020	0.706	0.021	-0.044	0.153	0.077
2.75	0.681	-0.020	0.682	0.015	-0.036	0.137	0.068
3.00	0.659	-0.019	0.660	0.010	-0.030	0.122	0.060
3.25	0.639	-0.017	0.640	0.007	-0.024	0.109	0.053
3.50	0.621	-0.014	0.622	0.005	-0.020	0.097	0.047
3.75	0.604	-0.012	0.605	0.003	-0.017	0.087	0.042
4.00	0.588	-0.010	0.589	0.002	-0.015	0.078	0.037
4.25	0.573	-0.008	0.574	0.002	-0.012	0.070	0.033
4.50	0.559	-0.006	0.560	0.001	-0.011	0.063	0.030
4.75	0.546	-0.005	0.547	0.001	-0.009	0.058	0.027
5.00	0.534	-0.004	0.534	0.001	-0.008	0.052	0.025

*in a composite disc of unit radius loaded by a unit tensile radial stress over the arc (-0.005,0.005) and by a concentrated reaction at the center. Layer thickness is 0.01.

PART V.

DEFORMATION COEFFICIENT OF FRICTION

If a load is applied to a portion of the surface of a body there will be an associated deformation. There will then be stored in the body strain energy which is equal to the work done by the load in deforming the body. We shall make certain assumptions regarding the behavior of this strain energy when a load is translated an infinitesimal distance on the surface. From these assumptions we will determine a friction force which leads to the idea of a deformation coefficient of friction. It must be emphasized that this deformation concept is not suggested as an alternative to the more commonly suggested mechanisms such as rupture of asperities and sticking. Rather it complements them as an additional possible mechanism.

Consider the case of a constant normal stress acting on a surface element of the disc. A neighboring state of loading is defined as a condition in which the load has been translated circumferentially an incremental distance, say δx . Clearly the amount of strain energy in the disc is unchanged as the system goes from one state of loading to a neighboring state. In a local sense, however, there has been a change. The radial displacement of each point on the surface has changed to that of its neighbor. Part of the disc has increased in strain energy while some has decreased. Specifically, a certain amount of work is done by the normal stress acting on that part of the loaded segment in which the magnitude of

the deformation increases. An equal amount of work is done by the deformed body over that part in which the deformation decreases.

The basis for the postulate of a deformation coefficient of friction stems from the assumption that the strain energy added to the system is supplied by the action of a circumferential force, while the strain energy released by the system is dissipated in the form of heat, noise, or other non-reversable action.

Although the model does not distinguish between sliding and rolling, the phenomenon is somewhat easier to visualize if we think of a rigid roller indenting a surface as it rolls. Work is required to deform the surface ahead of the roller. The energy released by the restoration of the initial configuration behind the roller is assumed lost. Simkins* suggests that interaction of molecules of the mating surfaces during sliding causes an additional dissipative effect not accounted for in this model. This interaction would be missing during rolling, however. He suggests, therefore, that the model is more realistic when thought of in terms of rolling contact.

To calculate the deformation coefficient we shall consider the elastic deformation of the surface due to a normal stress g. With reference to Figure 3, divide the loaded half-interval into n segments of width δx, where δx is the amount of the circumferential displacement. Associated with this displacement, the radial deformation of point x_i becomes u_{i-1}. The work required to deform the loaded surface in the i th interval is

*Thomas E. Simkins: Private communication.

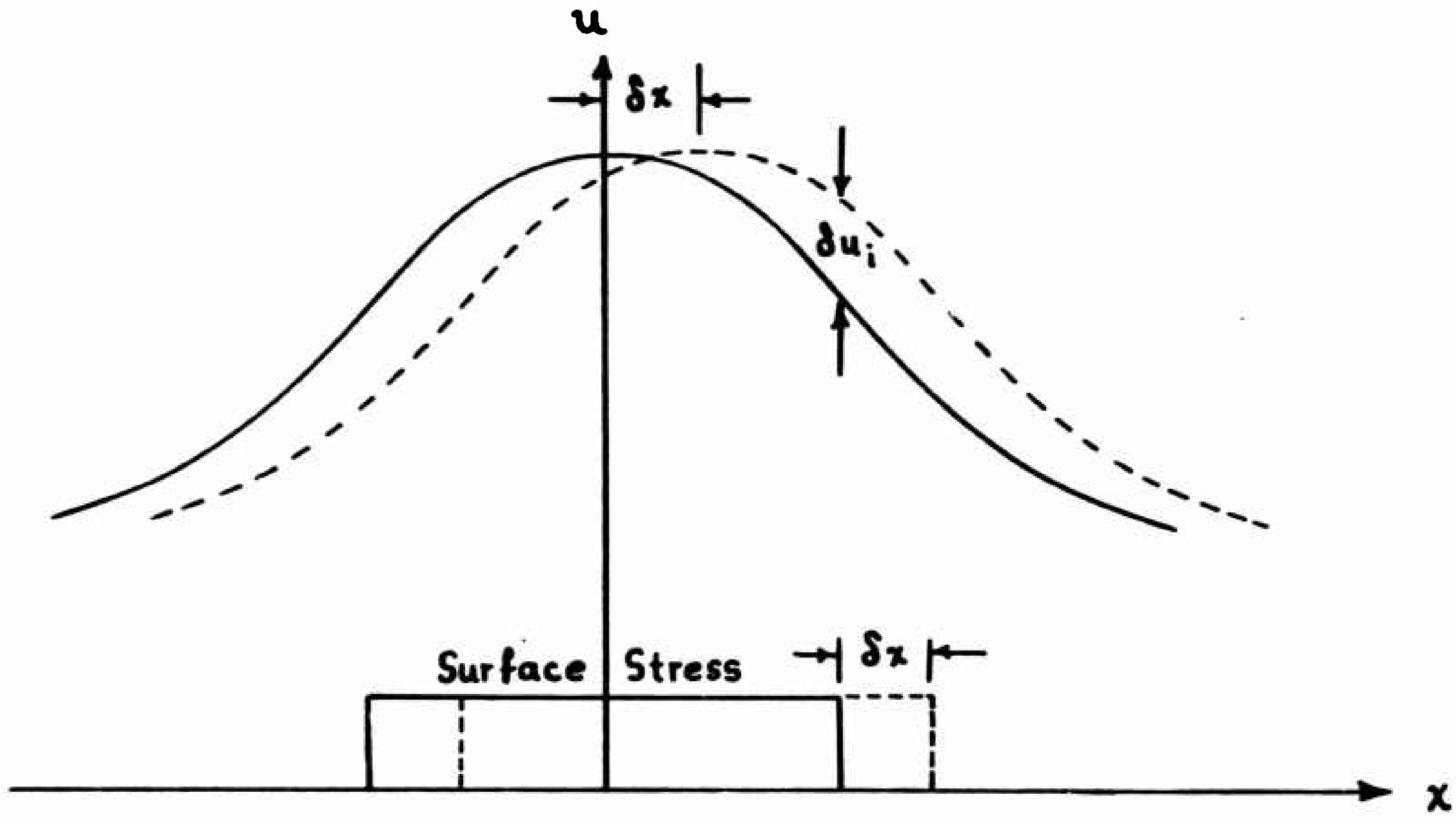

Figure 3. Incremental Translation of Surface Stress and Displacement

$$\delta W_i = g\ \delta x\left[\frac{u_{i-1}+u_i}{2} - \frac{u_i+u_{i+1}}{2}\right] \tag{90}$$

The expression for the last interval is one half of this. Summing over the pertinent half-range and simplifying gives

$$\delta W = \sum_{i=1}^{n} \delta W_i = g\ \delta x\left[\frac{u_0+u_1}{2} - \frac{u_{n-1}+2u_n+u_{n+1}}{4}\right] \tag{91}$$

Since δx is small and n is large, we may write

$$\delta W = g\ \delta x\,(u_0 - u_n) \tag{92}$$

The force P required to effect the translation must, by hypothesis, provide the energy δW. Therefore

$$P = \delta W/\delta x = g\,(u_0 - u_n) \tag{93}$$

In a disc of radius ρ loaded by a uniform normal stress g over an arc of width Φ, the total force F exerted by the load is

$$F = g\rho\Phi \tag{94}$$

The coefficient of friction Γ is defined similar to the usual form

$$\Gamma = \left|P/F\right| = \left|u_0 - u_n\right|/\rho\Phi \tag{95}$$

where the absolute value must be taken since the model does not distinguish between tensile and compressive forces. Since the deformation is directly proportional to the intensity g of the stress, the friction coefficient is a linear function of the normal force

applied over a constant element of arc.

To illustrate the computation, let us again use the caret to distinguish dimensional quantities. Then

$$\hat{u} = (g\rho/\hat{\mu})\,u(1,\theta) \tag{96}$$

and

$$\Gamma = \frac{u_0 - u_\pi}{\hat{\mu}\,\phi} \tag{97}$$

Applying the data for gold on steel as shown in Table II, we obtain

$$\Gamma = 3.4(10)^{-8}\,|g| \tag{98}$$

The friction coefficient due to the deformation is therefore quite small. For loads of moderate intensity, the effect is negligible compared to other friction mechanisms.

When the loaded arc width Φ is small the friction coefficient is essentially independent of Φ for a fixed ratio h/Φ. Results will be given in terms of the DFC, that is the coefficient which multiplies the stress to obtain the conventional coefficient of friction. Shown in Figure 4 is the behavior of the DFC for $\Phi = 0.01$, for both cases, gold on steel and alumina on steel. Table III lists some values of the normalized friction coefficient for these two cases.

Table III

DEFORMATION COEFFICIENT OF FRICTION

Part A. Gold Layer on Steel Base

DFC $\times (10)^8$ (in^2/lb) for prescribed Φ

h/Φ	Φ 1.0	0.1	0.01	0.001
0.0 (All Base)	1.23	1.45	1.47	1.47
0.005	1.24	1.46	1.49	-
0.01	1.25	1.47	1.49	-
0.02	1.27	1.49	1.51	-
0.05	1.33	1.56	1.58	-
0.1	1.43	1.67	1.69	1.70
0.2	1.65	1.90	1.92	1.93
0.5	2.64	2.70	2.71	2.71
1.0	3.31	3.40	3.40	3.40
2.0	-	3.71	3.73	3.73
5.0	-	3.81	3.84	3.85
10.	-	3.82	3.86	3.86
20.	-	-	3.86	3.87
50.	-	-	3.86	3.87
100.	-	-	3.86	3.87
200.	-	-	-	3.87
500.	-	-	-	3.87
(All Layer)	3.31	3.82	3.86	3.87

Table III

DEFORMATION COEFFICIENT OF FRICTION

Part B. Alumina Layer on Steel Base

DFC $\times (10)^8$ (in^2/lb) for prescribed Φ

h/Φ	Φ 1.0	0.1	0.01	0.001
0.0 (All Base)	1.23	1.45	1.47	1.47
0.005	1.22	1.44	-	-
0.01	1.22	1.44	1.46	-
0.02	1.21	1.42	1.45	-
0.05	1.18	1.40	1.42	-
0.1	1.15	1.36	1.38	-
0.2	1.06	1.26	1.28	-
0.5	0.72	0.97	0.99	0.99
1.0	0.56	0.78	0.80	0.80
2.0	-	0.70	0.72	0.72
5.0	-	0.68	0.69	0.70
10.	-	0.68	0.69	0.69
20.	-	-	0.69	0.69
50.	-	-	0.69	0.69
100.	-	-	0.69	0.69
200.	-	-	-	0.69
500.	-	-	-	0.69
(All Layer)	0.56	0.68	0.69	0.69

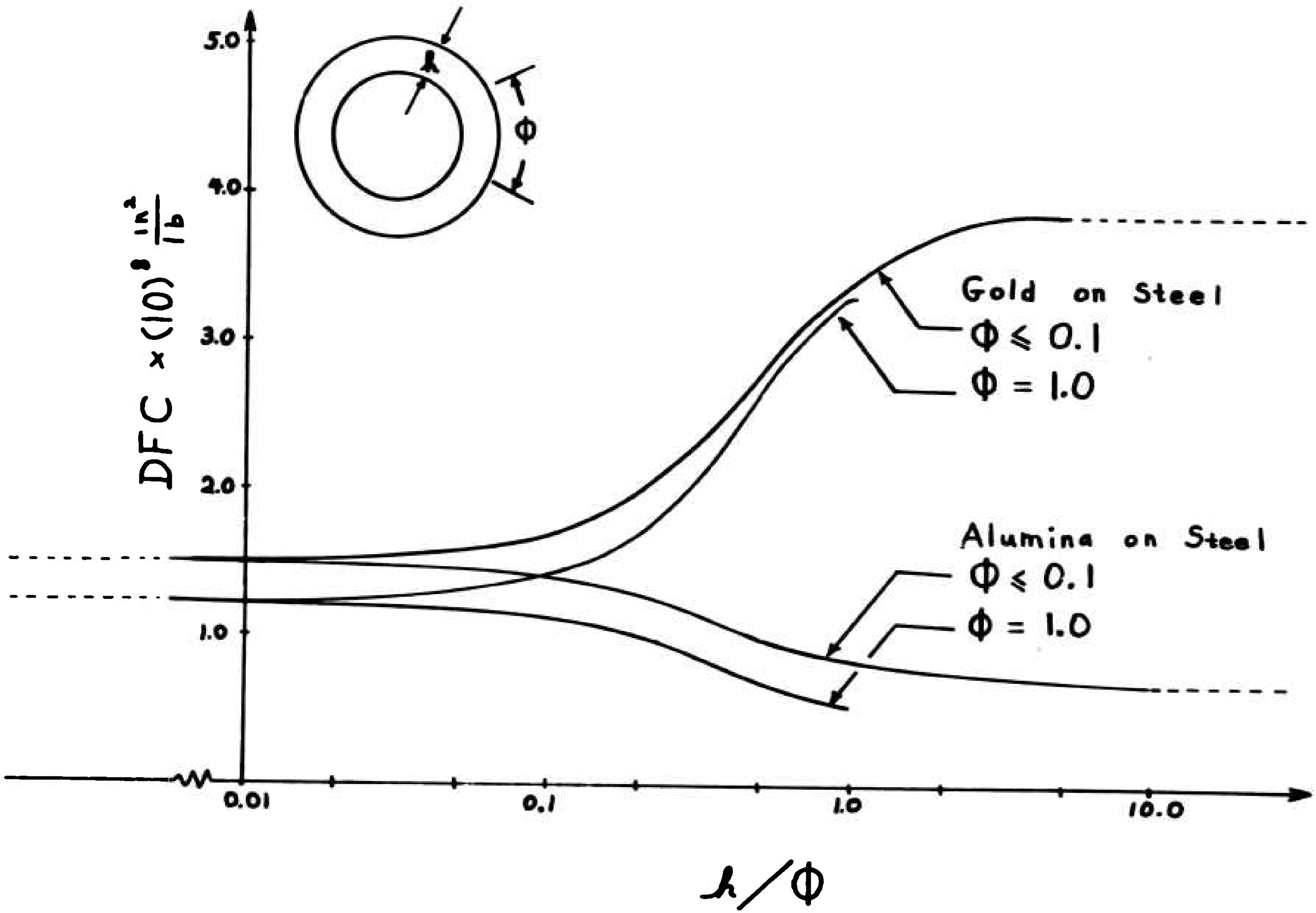

Figure 4. Deformation Coefficient of Friction

PART VI.

CONCLUSIONS

The finite Fourier transform has been used to develop a field solution for a composite disc in a state of plane elastic equilibrium. The method has the advantage of being explicit compared to more familiar methods. No prior knowledge of the solution is required. On the other hand it has the disadvantage of being somewhat cumbersome, and it does not make use of the accumulated collections of known solutions to the biharmonic problem.

The properties of the finite Fourier transform cause the method to have merit for circular regions with all boundarys concentric circles. The problem examined in this investigation was a fundamental elasticity problem of the first kind. If displacements were prescribed instead of stresses the technique should be equally useful. In the event of mixed boundary conditions, however, the method may lead to difficulties associated with transforming the boundary conditions.

In the present problem the transform of the pair of partial differential equations in two independent displacement variables reduces to a pair of ordinary differential equations. These are solved explicitly. The inversion of the transformed solution is obtained by summing the series of solutions to the transformed problem.

The numerical evaluation of the sum was accomplished with some difficulty. In most of the cases of interest, direct evaluation could not be performed. Convergence was obtained, however, by using the closed form solution for a homogeneous disc as a first approximation. Even then, convergence is slow. Because an increasingly large number

of terms is required as the loaded sector becomes small, there may be an accumulated rounding error for some results. Double precision computation could eliminate some of these. If results are required only in the vicinity of the load, the solution to the problem of a semi-infinite plate would be adequate for the small loaded sector where curvature effects are negligible. It was found that for $\Phi \leqslant 0.1$ one could essentially disregard curvature effects.

A postulated mechanism for friction on a smooth surface was developed. Friction results from assumptions about the dissipation of strain energy when a deformed body is transformed to a neighboring deformation state. The friction coefficient found by this method is very small; its presence would be obscured by more significant factors. However, this frictional contribution could be a limitation on the complete elimination of friction between solid surfaces. To the extent that this theory is valid, bodies with a high elastic modulus will exhibit less friction than one with a low modulus due to the smaller amount of strain energy associated with a given load.

PART VII.

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13. ABSTRACT

This investigation is directed to the analysis of a composite disc in plane elastic equilibrium. The disc consists of concentric regions, each of homogeneous properties, and is loaded on its outer surface by an arbitrary load. Reactions at the center are prescribed to provide the necessary equilibrating force and moment. The equations of equilibrium are specified in terms of the radial and tangential displacements, giving a pair of partial differential in plane polar coordinates.

A finite Fourier transform is applied to reduce the partial differential equations to a pair of ordinary differential equations with the radial coordinate as independent variables. Inversion of the transformed solution yields an infinite series. The analysis is continued in further detail for uniformly distributed and concentrated normal and shear loads.

The deformations associated with the uniformly distributed normal load are used as the basis of a postulated friction phenomenon called the deformation coefficient of friction. Assumptions about the behavior of the strain energy as a load is translated an incremental distance give rise to a friction force, which in turn permits a friction coefficient to be calculated. This coefficient is characterized by being proportional to the load rather than constant. The contribution of the deformation coefficient to the total friction behavior is ordinarily quite small.

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